

Chapter 2 Diffusion in dilute solutions

1. Water evaporation

To find the rate of evaporation, we need to find the flux of water across the air film:

$$j_1 = \frac{DH}{\ell} \Delta c_1$$

Since the film is made up of air, $H = 1$. We are given D and ℓ , so we need to find Δc_1 . We first calculate the difference in the partial pressure of water across the film. From a steam table we find that at 20 °C, $p_{sat} = 2.3388$ kPa. Since the air immediately above the water is saturated we have:

$$\Delta p_1 = p_{sat} - 0.5 p_{sat} = 0.5 p_{sat} = 0.5(2.3388 \text{ kPa}) = 1.1694 \text{ kPa} = 1169.4 \text{ Pa}$$

Assuming an ideal gas, we can find the concentration difference:

$$\Delta c_1 = \frac{\Delta p_1}{RT} = \frac{1169.4 \text{ Pa}}{\left(8.3145 \frac{\text{J}}{\text{K} \cdot \text{mol}}\right)(293 \text{ K})} = 0.48 \frac{\text{mol}}{\text{m}^3} = 4.8 \cdot 10^{-7} \frac{\text{mol}}{\text{cm}^3}$$

We can now calculate the flux:

$$j_1 = \frac{0.25 \frac{\text{cm}^2}{\text{s}}}{0.15 \text{ cm}} \left(4.8 \cdot 10^{-7} \frac{\text{mol}}{\text{cm}^3}\right) = 8.0 \cdot 10^{-7} \frac{\text{mol}}{\text{cm}^2 \text{ s}}$$

To find the height change, we divide by the molar density of water:

$$\frac{8.0 \cdot 10^{-7} \frac{\text{mol}}{\text{cm}^2 \text{ s}}}{1 \frac{\text{g}}{\text{cm}^3} \cdot \frac{1 \text{ mol}}{18.015 \text{ g}}} = 1.44 \cdot 10^{-5} \frac{\text{cm}}{\text{s}} = 1.25 \frac{\text{cm}}{\text{day}}$$

2. Diffusion across a monolayer

Recalling that the resistance is the inverse of the permeance, we have:

$$\frac{1}{P} = 2 \frac{\text{s}}{\text{cm}} \Rightarrow P = 0.5 \frac{\text{cm}}{\text{s}}$$

We know that the permeance is given by:

$$P = \frac{DH}{\ell} \Rightarrow D = \frac{P\ell}{H} = \frac{\left(0.5 \frac{\text{cm}}{\text{s}}\right)(2.5 \cdot 10^{-7} \text{ cm})}{0.018} = 7 \cdot 10^{-6} \frac{\text{cm}^2}{\text{s}}$$

3. Diffusion coefficient of NO₂ in water

We treat the water as a semiinfinite slab. From Eq. (2.3-18) we know that the flux at the surface is given by:

$$j_1|_{z=0} = \sqrt{\frac{D}{\pi t}} (c_{10} - c_{1\infty})$$

The concentration at the surface is given by Henry's Law:

$$c_{10} = \frac{p}{H} = \frac{0.93 \text{ atm}}{37000 \frac{\text{cm}^3 \text{ atm}}{\text{mol}}} = 2.5 \cdot 10^{-5} \frac{\text{mol}}{\text{cm}^3}$$

Since we have a semiinfinite slab we assume $c_{1\infty} = 0$. To find the flux over the total time t , we integrate:

$$N_1 = \int_0^t A j_1|_{z=0} dt = A \int_0^t \sqrt{\frac{D}{\pi t}} (c_{10} - 0) dt = A c_{10} \sqrt{\frac{D}{\pi}} \int_0^t \frac{dt}{\sqrt{t}} = 2 A c_{10} \sqrt{\frac{D}{\pi}} [\sqrt{t}]_0^t = 2 A c_{10} \sqrt{\frac{D t}{\pi}}$$

Assuming an ideal gas, the flux N_1 is given by:

$$N_1 = \frac{PV}{RT} = \frac{(0.93 \text{ atm})(0.82 \text{ cm}^3)}{\left(82.06 \frac{\text{cm}^3 \text{ atm}}{\text{K} \cdot \text{mol}}\right)(289 \text{ K})} = 3.2 \cdot 10^{-5} \text{ mol}$$

Solving for D , we have:

$$D = \left(\frac{N_1}{2 A c_{10}} \right)^2 \cdot \frac{\pi}{t} = \left[\frac{3.2 \cdot 10^{-5} \text{ mol}}{(36.3 \text{ cm}^2) \left(2.5 \cdot 10^{-5} \frac{\text{mol}}{\text{cm}^3} \right)} \right]^2 \cdot \frac{\pi}{180 \text{ s}} = 5.4 \cdot 10^{-6} \frac{\text{cm}^2}{\text{s}}$$

4. Permeability of water across a polymer film

The rate of water loss can be found by linear regression or by simply using the first and last data points:

$$N_1 = \frac{14.0153 \text{ g} - 13.5256 \text{ g}}{16 \text{ days}} = 0.031 \frac{\text{g}}{\text{day}}$$

The molar flux is given by:

$$j_1 = \frac{N_1}{A} = \frac{0.031 \frac{\text{g}}{\text{day}} \cdot \frac{1 \text{ mol}}{18.015 \text{ g}} \cdot \frac{1 \text{ day}}{24 \text{ hr}} \cdot \frac{1 \text{ hr}}{3600 \text{ s}}}{85.6 \text{ cm}^2} = 2.3 \cdot 10^{-10} \frac{\text{mol}}{\text{cm}^2 \text{ s}}$$

From Eq. (2-2.10) we have:

$$j_1 = \frac{DH}{\ell} \Delta c_1$$

We need to find Δc_1 . We first calculate the difference in the partial pressure of water across the film. From a steam table we find that at 35 °C, $p_{\text{sat}} = 0.0555 \text{ atm}$. Since the air inside the bag is saturated, we have:

$$\Delta p_1 = p_{\text{sat}} - 0.75 p_{\text{sat}} = 0.25 p_{\text{sat}} = 0.25(0.0555 \text{ atm}) = 0.0139 \text{ atm}$$

Assuming an ideal gas, the concentration difference is:

$$\Delta c_1 = \frac{\Delta p_1}{RT} = \frac{0.0139 \text{ atm}}{\left(82.06 \frac{\text{cm}^3 \text{ atm}}{\text{K} \cdot \text{mol}}\right)(308 \text{ K})} = 5.49 \cdot 10^{-7} \frac{\text{mol}}{\text{cm}^3}$$

The permeability DH is given by:

$$DH = \frac{j_1 \ell}{\Delta c_1} = \frac{\left(2.3 \cdot 10^{-10} \frac{\text{mol}}{\text{cm}^2 \text{ s}}\right)(0.051 \text{ cm})}{5.49 \cdot 10^{-7} \frac{\text{mol}}{\text{cm}^3}} = 2.1 \cdot 10^{-5} \frac{\text{cm}^2}{\text{s}}$$

5. Diaphragm cell

From Ex. (2.2-4) we have:

$$D = \frac{1}{\beta t} \ln \left(\frac{\Delta c_{10}}{\Delta c_1} \right) \quad \beta = \frac{AH}{\ell} \left(\frac{1}{V_{\text{lower}}} + \frac{1}{V_{\text{upper}}} \right)$$

(a) Using the first equation and solving for β we have:

$$\beta = \frac{1}{Dt} \ln \left(\frac{\Delta c_{10}}{\Delta c_1} \right) = \frac{1}{\left(1.859 \cdot 10^{-5} \frac{\text{cm}^2}{\text{s}}\right) \left(36 \text{ hr} \cdot \frac{3600 \text{ s}}{\text{hr}} + 6 \text{ min} \cdot \frac{60 \text{ s}}{\text{min}}\right)} \ln \left(\frac{1}{0.492} \right)$$

$$= 0.294 \text{ cm}^{-2}$$

(b) Using the second equation and solving for ℓ we have:

$$\ell = \frac{AH}{\beta} \left(\frac{1}{V_{\text{lower}}} + \frac{1}{V_{\text{upper}}} \right) = \frac{\left[\frac{\pi}{4} (2.51 \text{ cm})^2 \right] (0.34)}{0.294 \text{ cm}^{-2}} \left(\frac{1}{42.3 \text{ cm}^3} + \frac{1}{40.8 \text{ cm}^3} \right) = 0.276 \text{ cm}$$

(c) Since the porosity is the same, the result is the same.

6. Measuring diffusion coefficient of gases

The maximum concentration will be at the center of the pipe so $r = 0$ and the equation reduces to:

$$c_1 = \frac{Q}{4\pi Dz}$$

Dimensional analysis tells us that the concentration must be dimensionless, so we simply convert it from a weight fraction w_1 to a mole fraction y_1 :

$$w_1 = \frac{4y_1}{4y_1 + 28(1-y_1)} \Rightarrow \frac{1}{w_1} = \frac{7}{y_1} - 6 \Rightarrow y_1 = \frac{7}{\frac{1}{w_1} + 6} = \frac{7}{\frac{1}{0.0048} + 6} = 0.033 = c_1$$

Solving for D , we have:

$$D = \frac{Q}{4\pi c_1 z} = \frac{0.045 \frac{\text{cm}^3}{\text{s}}}{4\pi (0.033)(1.031 \text{ cm})} = 0.11 \frac{\text{cm}^2}{\text{s}}$$

7. Carburizing of steel

Rearranging Eq. (2.3-15) we have:

$$c_1 = c_{10} - (c_{1\infty} - c_{10}) \operatorname{erf}\left(\frac{z}{\sqrt{4Dt}}\right)$$

Since this is a semiinfinite slab, $c_{1\infty} = 0$. Assuming the argument of the error function is small (i.e. less than about 0.7), we can approximate it as the argument itself:

$$c_1 \approx c_{10} - \frac{c_{10}z}{\sqrt{4Dt}}$$

Using the data for 10 hr we calculate the slope from its endpoints:

$$m = \frac{0.7 - 1.35}{(0.05 - 0) \text{ in} \cdot \frac{2.54 \text{ cm}}{\text{in}}} = 5.12 \text{ cm}^{-1}$$

The data show that the concentration c_1 at 10 hr is given by:

$$c_1 = 1.35 - 5.12z$$

By comparison with our approximation we have:

$$c_{10} = 1.35$$

$$\frac{c_{10}}{\sqrt{4Dt}} = 5.12$$

Solving for D gives:

$$D = \frac{\left(\frac{c_{10}}{5.12}\right)^2}{4t} = \frac{\left(\frac{1.35}{5.12 \text{ cm}^{-1}}\right)^2}{4\left(10 \text{ hr} \cdot \frac{3600 \text{ s}}{\text{hr}}\right)} = 4.8 \cdot 10^{-7} \frac{\text{cm}^2}{\text{s}}$$

To check whether our approximation was valid, we calculate the argument of the error function:

$$\frac{z}{\sqrt{4Dt}} = \frac{0.05 \text{ in} \cdot \frac{2.54 \text{ cm}}{\text{in}}}{\sqrt{4\left(4.8 \cdot 10^{-7} \frac{\text{cm}^2}{\text{s}}\right)\left(10 \text{ hr} \cdot \frac{3600 \text{ s}}{\text{hr}}\right)}} = 0.48$$

The assumption is valid.

8. Twin-bulb method

We define the direction of positive flux is from bulb A into bulb B. A balance on bulb A gives:

$$V \frac{dc_{1A}}{dt} = -(\pi r^2) j_1$$

Similarly for bulb B we have:

$$V \frac{dc_{1B}}{dt} = (\pi r^2) j_1$$

Subtracting the second equation from the first, we get:

$$V \frac{d}{dt}(c_{1A} - c_{1B}) = -2\pi r^2 j_1$$

The flux is given by:

$$j_1 = \frac{DH}{\ell}(c_{1A} - c_{1B})$$

In this case $H = 1$ since there is no interface. Combining these equations and substituting Δc_1 for $c_{1A} - c_{1B}$ we have:

$$V \frac{d\Delta c_1}{dt} = -\frac{2\pi D \Delta c_1 r^2}{\ell} \Rightarrow \frac{d\Delta c_1}{\Delta c_1} = -\frac{2\pi D r^2}{\ell V} dt$$

Integration gives:

$$\int_{\Delta c_{10}}^{\Delta c_1} \frac{d\Delta c_1}{\Delta c_1} = -\frac{2\pi D r^2}{\ell V} \int_0^t dt \Rightarrow \ln\left(\frac{\Delta c_1}{\Delta c_{10}}\right) = -\frac{2\pi D r^2 t}{\ell V} \Rightarrow \Delta c_1 = \Delta c_{10} e^{-\frac{2\pi D r^2 t}{\ell V}}$$

9. Steady-state flux out of a pipe with porous wall

From Eq. (2.4-29) we have:

$$\frac{\partial c_1}{\partial t} = \frac{D}{r} \frac{\partial}{\partial r} \left(r \frac{\partial c_1}{\partial r} \right)$$

Since we are at steady state, the time derivative is zero and we are left with:

$$0 = \frac{D}{r} \frac{d}{dr} \left(r \frac{dc_1}{dr} \right) \Rightarrow \frac{d}{dr} \left(r \frac{dc_1}{dr} \right) = 0$$

Integrating twice we have:

$$r \frac{dc_1}{dr} = A \Rightarrow \int dc_1 = A \int \frac{dr}{r} \Rightarrow c_1 = A \ln r + B$$

This is subject to boundary conditions:

$$r = R_i \quad c = c_{1i}$$

$$r = R_0 \quad c = 0$$

Applying the second and then the first we have:

$$0 = A \ln R_0 + B \Rightarrow B = -A \ln R_0$$

$$c_{1i} = A \ln R_i + B = A \ln R_i - A \ln R_0 = A \ln \frac{R_i}{R_0} \Rightarrow A = \frac{c_{1i}}{\ln \frac{R_i}{R_0}}$$

$$B = -A \ln R_0 = -\frac{c_{1i} \ln R_0}{\ln \frac{R_i}{R_0}}$$

Substitution gives:

$$c_1 = \frac{c_{1i} \ln r}{\ln \frac{R_i}{R_o}} - \frac{c_{1i} \ln R_o}{\ln \frac{R_i}{R_o}} = c_{1i} \frac{\ln \frac{r}{R_o}}{\ln \frac{R_i}{R_o}}$$

The flux at the outside of the pipe is given by Fick's Law:

$$j = -D \frac{dc_1}{dr} \Big|_{r=R_o} = - \frac{Dc_{1i}}{r \ln \frac{R_i}{R_o}} \Big|_{r=R_o} = \frac{Dc_{1i}}{R_o \ln \frac{R_o}{R_i}}$$

10. Controlled release of pheromones

From a balance on the device we have:

$$V \frac{dc_1}{dt} = r_o - Aj_1$$

Since we are interested in the steady state case, the time derivative is 0 and we find that the rate of sublimation must equal the rate of transport through the membrane. Substituting for r_o and j_1 we have:

$$r_o = Aj_1 \Rightarrow 6 \cdot 10^{-17} \left[1 - \left(1.1 \cdot 10^7 \frac{\text{cm}^3}{\text{mol}} \right) c_1 \right] \frac{\text{mol}}{\text{s}} = \frac{ADH}{\ell} (c_1 - 0)$$

(a) Solving for c_1 we have:

$$\begin{aligned} c_1 &= \frac{6 \cdot 10^{-17} \frac{\text{mol}}{\text{s}}}{\frac{ADH}{\ell} + 6.6 \cdot 10^{-10} \frac{\text{cm}^3}{\text{s}}} = \frac{6 \cdot 10^{-17} \frac{\text{mol}}{\text{s}}}{\frac{(1.8 \text{ cm}^2) \left(1.92 \cdot 10^{-12} \frac{\text{cm}^2}{\text{s}} \right)}{0.06 \text{ cm}} + 6.6 \cdot 10^{-10} \frac{\text{cm}^3}{\text{s}}} \\ &= 8.4 \cdot 10^{-8} \frac{\text{mol}}{\text{cm}^3} \end{aligned}$$

(b) Solving for r_o or Aj_1 gives:

$$Aj_1 = \frac{ADH}{\ell} c_1 = \frac{(1.8 \text{ cm}^2) \left(1.92 \cdot 10^{-12} \frac{\text{cm}^2}{\text{s}} \right)}{0.06 \text{ cm}} \left(8.4 \cdot 10^{-8} \frac{\text{mol}}{\text{cm}^3} \right) = 4.8 \cdot 10^{-18} \frac{\text{mol}}{\text{s}}$$

11. Measuring age of antique glass

Based on Example 2.3-3, we have:

$$j_1|_{z=0} = \sqrt{\frac{D(1+K)}{\pi t}} (c_{10} - c_{1\infty})$$

Since the water is consumed as it enters the glass, we assume that $c_{1\infty} = 0$. The total hydration is

then:

$$N_1 = A \int_0^t j_1|_{z=0} dt = A c_{10} \sqrt{\frac{D(1+K)}{\pi}} \int_0^t \frac{dt}{\sqrt{t}} = 2 A c_{10} \sqrt{\frac{Dt(1+K)}{\pi}}$$

Solving for t we have:

$$t = \frac{\pi}{D(1+K)} \left(\frac{N_1}{2 A c_{10}} \right)^2$$

12. Diffusion in a reactive barrier membrane

(a) Mass balances on mobile species 1 and immobile species 2 give:

$$\frac{\partial c_1}{\partial t} = D \frac{\partial^2 c_1}{\partial z^2} - k_R c_1 c_2$$

$$\frac{\partial c_2}{\partial t} = -k_R c_1 c_2$$

(b) The boundary conditions for this situation are:

$$\begin{array}{llll} t < 0 & \text{all } z & c_1 = 0 & c_2 = c_{20} \\ t \geq 0 & z = 0 & c_1 = H c_{10} & \frac{\partial c_2}{\partial z} = 0 \\ & z = \ell & c_1 = 0 & \frac{\partial c_2}{\partial z} = 0 \end{array}$$

(c) The reaction term would exist only in a front, moving across the film with time. Everywhere else in the film either c_1 or c_2 is zero and the reaction term drops out of the differential equations.

13. Diffusion of dopant in arsenide

From Eq. (2.4-14) we have:

$$c_1 = \frac{M/A}{\sqrt{4\pi Dt}} e^{-\frac{z^2}{4Dt}}$$

Since the maximum concentration will be at $z = 0$ i.e. the site of the scratch we can write:

$$c_{1\max} = \frac{M/A}{\sqrt{4\pi Dt}} \Rightarrow \frac{c_1}{c_{1\max}} = e^{-\frac{z^2}{4Dt}}$$

Solving for t , we have:

$$t = \frac{z^2}{4D \ln\left(\frac{c_{1\max}}{c_1}\right)} = \frac{(4 \cdot 10^{-4} \text{ cm})^2}{4 \left(10^{-11} \frac{\text{cm}^2}{\text{s}} \right) \ln(10)} = 1740 \text{ s} = 29 \text{ min}$$

14. Concentration profile in Fick's experiment

(a) For the cylinder, the cross sectional area A is constant. Since the experiment is at steady state, j_1 is also constant. Using Fick's Law, we have:

$$j = -D \frac{dc_1}{dz} = B \Rightarrow c_1 = -\frac{B}{D} z + C$$

The boundary conditions are:

$$z = 0 \quad c_1 = 0$$

$$z = \ell \quad c_1 = c_{1sat}$$

The first boundary condition tells us that $C = 0$. Applying the second gives:

$$c_{1sat} = -\frac{B}{D} \ell \Rightarrow B = -\frac{c_{1sat} D}{\ell} \Rightarrow c_1 = c_{1sat} \frac{z}{\ell}$$

(b) For the funnel, the cross sectional area A is not constant. The area at height z is given by:

$$A = \pi \left(R_0 + \frac{R_\ell - R_0}{\ell} z \right)^2 = \pi (R_0 + kz)^2 \quad k = \frac{R_\ell - R_0}{\ell}$$

where k has been defined for convenience. Since the area is a function of z , the condition of steady state requires that the product Aj_1 be constant. As a result we have:

$$0 = \frac{d(Aj_1)}{dz} = j_1 \frac{dA}{dz} + A \frac{dj_1}{dz}$$

Using Fick's Law to substitute for j_1 and the above expression for A we have:

$$0 = \left[-D \frac{dc_1}{dz} \right] [2\pi k (R_0 + kz)] + \left[\pi (R_0 + kz)^2 \right] \left[-D \frac{d^2 c_1}{dz^2} \right]$$

$$0 = 2k \frac{dc_1}{dz} + (R_0 + kz) \frac{d^2 c_1}{dz^2}$$

We now define a function u and redefine our differentials in terms of it:

$$u = R_0 + kz$$

$$\frac{dc_1}{dz} = \frac{dc_1}{du} \frac{du}{dz} = k \frac{dc_1}{du}$$

$$\frac{d^2 c_1}{dz^2} = \frac{d^2 c_1}{du^2} \left(\frac{du}{dz} \right)^2 = k^2 \frac{d^2 c_1}{du^2}$$

Our equation now becomes:

$$0 = 2k^2 \frac{dc_1}{du} + k^2 u \frac{d^2 c_1}{du^2} \Rightarrow u^2 \frac{d^2 c_1}{du^2} + 2u \frac{dc_1}{du} = 0$$

This equation is of the Euler-Cauchy form, which means the solution is of the form

$$c_1 = u^m:$$

$$u^2 [m(m-1)u^{m-2}] + 2u(mu^{m-1}) = 0 \Rightarrow m^2 + m = m(m+1) = 0 \Rightarrow m = 0, -1$$

The solution is a linear combination of the two roots:

$$c_1 = B + \frac{C}{u} = B + \frac{C}{R_0 + kz}$$

Applying the same boundary conditions as above we find:

$$0 = B + \frac{C}{R_0} \Rightarrow B = -\frac{C}{R_0}$$

$$c_{1sat} = \frac{C}{R_0 + k\ell} - \frac{C}{R_0} \Rightarrow C = \frac{c_{1sat}}{\frac{1}{R_0 + k\ell} - \frac{1}{R_0}} = \frac{c_{1sat} R_0 (R_0 + k\ell)}{k\ell}$$

The concentration profile is therefore:

$$c_1 = \frac{c_{1sat} R_0 (R_0 + k\ell)}{k\ell} \left[\frac{1}{R_0 + kz} - \frac{1}{R_0} \right] = c_{1sat} \frac{R_0 + k\ell}{R_0 + kz} \frac{z}{\ell} = c_{1sat} \frac{R_\ell z}{R_0 \ell + z(R_\ell - R_0)}$$

15. Bacteria between membranes

(a) From Fick's Law we have:

$$j_s = -D \frac{dc_s}{dz}$$

At steady state we know that j_s is independent of z :

$$\frac{dj_s}{dz} = 0 = -D \frac{d^2 c_s}{dz^2} \quad \begin{array}{ll} z = 0 & c_s = c_{s0} \\ z = \ell & c_s = 0 \end{array}$$

By applying the boundary conditions and integrating twice we have:

$$j_s = \frac{D}{\ell} c_{s0}$$

By comparison with Fick's Law we can write:

$$-D \frac{dc_s}{dz} = \frac{D}{\ell} c_{s0} \Rightarrow \int_{c_{s0}}^{c_s} dc_s = -\frac{c_{s0}}{\ell} \int_0^z dz \Rightarrow c_s = c_{s0} \left(1 - \frac{z}{\ell} \right)$$

(b) Once again assuming steady state, we know that j_B is independent of z . We also know from the boundary conditions that $j_B = 0$ at $z = 0$ and $z = \ell$, so $j_B = 0$ for all z . The given equation then becomes:

$$0 = -D_0 \frac{dc_B}{dz} + \chi c_B \frac{dc_s}{dz} \Rightarrow D_0 \frac{dc_B}{dz} = \chi c_B \left(-\frac{c_{s0}}{\ell} \right)$$

Integration gives:

$$\int_{c_{B0}}^{c_B} \frac{dc_B}{c_B} = -\frac{\chi c_{s0}}{D_0 \ell} \int_0^z dz \Rightarrow \ln \frac{c_B}{c_{B0}} = -\frac{\chi c_{s0} z}{D_0 \ell} \Rightarrow c_B = c_{B0} e^{-\frac{\chi c_{s0} z}{D_0 \ell}}$$

where c_{B0} is the concentration of B at $z = 0$.

16. Extraction of sucrose

From Eq. (2-3.18) the flux through the surface of a slice is:

$$j_1 = \sqrt{\frac{D}{\pi t}} c_{10}$$

The total flux is a weighted average of the individual fluxes:

$$j_1 = \varepsilon j_c + (1 - \varepsilon)j_w = \varepsilon \sqrt{\frac{D_c}{\pi t}} c_{10} + (1 - \varepsilon) \sqrt{\frac{D_w}{\pi t}} c_{10}$$

Integrating over time to get the total sucrose extracted per unit area we have:

$$M = \int_0^t j_1 dt = \frac{c_{10}}{\sqrt{\pi}} \int_0^t \varepsilon \sqrt{\frac{D_c}{t}} + (1 - \varepsilon) \sqrt{\frac{D_w}{t}} dt = 2c_{10} \sqrt{\frac{t}{\pi}} \left[\varepsilon \sqrt{D_c} + (1 - \varepsilon) \sqrt{D_w} \right]$$

From the given expression we have:

$$D = \frac{\pi}{4t} \left(\frac{M}{c_{10}} \right)^2 = \frac{\pi}{4t} \left(\frac{2c_{10} \sqrt{\frac{t}{\pi}} \left[\varepsilon \sqrt{D_c} + (1 - \varepsilon) \sqrt{D_w} \right]}{c_{10}} \right)^2 = \left[\varepsilon \sqrt{D_c} + (1 - \varepsilon) \sqrt{D_w} \right]^2$$

Not sure this answers the question