# CHAPTER 1
# Rosenblatt's Perceptron

## Problem 1.1

(1) If $\mathbf{w}^T(n)\mathbf{x}(n) > 0$, then $y(n) = +1$.
If also $\mathbf{x}(n)$ belongs to $C_1$, then $d(n) = +1$.
Under these conditions, the error signal is

$$e(n) = d(n) - y(n) = 0$$

and from Eq. (1.22) of the text:

$$\mathbf{w}(n+1) = \mathbf{w}(n) + \eta e(n)\mathbf{x}(n) = \mathbf{w}(n)$$

This result is the same as line 1 of Eq. (1.5) of the text.

(2) If $\mathbf{w}^T(n)\mathbf{x}(n) < 0$, then $y(n) = -1$.
If also $\mathbf{x}(n)$ belongs to $C_2$, then $d(n) = -1$.
Under these conditions, the error signal $e(n)$ remains zero, and so from Eq. (1.22) we have

$$\mathbf{w}(n+1) = \mathbf{w}(n)$$

This result is the same as line 2 of Eq. (1.5).

(3) If $\mathbf{w}^T(n)\mathbf{x}(n) > 0$ and $\mathbf{x}(n)$ belongs to $C_2$ we have

$$y(n) = +1$$
$$d(n) = -1$$

The error signal $e(n)$ is -2, and so Eq. (1.22) yields

$$\mathbf{w}(n+1) = \mathbf{w}(n) - 2\eta\mathbf{x}(n)$$

which has the same form as the first line of Eq. (1.6), except for the scaling factor 2.

(4) Finally if $\mathbf{w}^T(n)\mathbf{x}(n) < 0$ and $\mathbf{x}(n)$ belongs to $C_1$, then

$$y(n) = -1$$
$$d(n) = +1$$

In this case, the use of Eq. (1.22) yields

$$\mathbf{w}(n+1) = \mathbf{w}(n) + 2\eta\mathbf{x}(n)$$

which has the same mathematical form as line 2 of Eq. (1.6), except for the scaling factor 2.

## Problem 1.2

The output signal is defined by

$$\begin{aligned} y &= \tanh\left(\frac{v}{2}\right) \\ &= \tanh\left(\frac{b}{2} + \frac{1}{2}\sum_i w_i x_i\right) \end{aligned}$$

Equivalently, we may write

$$b + \sum_i w_i x_i = y' \tag{1}$$

where

$$y' = 2\tanh^{-1}(y)$$

Equation (1) is the equation of a hyperplane.

**Problem 1.3**

(a) AND operation: Truth Table 1

| Inputs | | Output |
|---|---|---|
| $x_1$ | $x_2$ | y |
| 1 | 1 | 1 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 0 | 0 | 0 |

This operation may be realized using the perceptron of Fig. 1

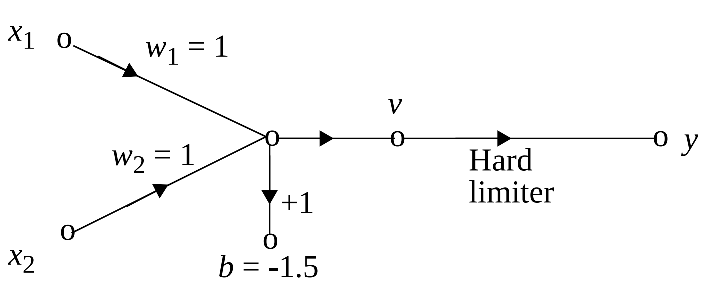


Figure 1: Problem 1.3

The hard limiter input is

$$\begin{aligned} v &= w_1 x_1 + w_2 x_2 + b \\ &= x_1 + x_2 - 1.5 \end{aligned}$$

If $x_1 = x_2 = 1$, then $v = 0.5$, and $y = 1$
If $x_1 = 0$, and $x_2 = 1$, then $v = -0.5$, and $y = 0$
If $x_1 = 1$, and $x_2 = 0$, then $v = -0.5$, and $y = 0$
If $x_1 = x_2 = 0$, then $v = -1.5$, and $y = 0$

These conditions agree with truth table 1.

OR operation: Truth Table 2

| Inputs | | Output |
|---|---|---|
| $x_1$ | $x_2$ | y |
| 1 | 1 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 0 | 0 | 0 |

The OR operation may be realized using the perceptron of Fig. 2:

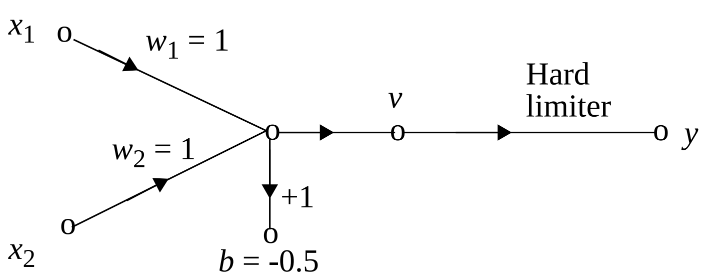


Figure 2: Problem 1.3

In this case, the hard limiter input is

$$v = x_1 + x_2 - 0.5$$

If $x_1 = x_2 = 1$, then $v = 1.5$, and $y = 1$
If $x_1 = 0$, and $x_2 = 1$, then $v = 0.5$, and $y = 1$
If $x_1 = 1$, and $x_2 = 0$, then $v = 0.5$, and $y = 1$
If $x_1 = x_2 = 0$, then $v = -0.5$, and $y = -1$

These conditions agree with truth table 2.

COMPLEMENT operation: Truth Table 3

| Input $x$, | Output, y |
|---|---|
| 1 | 0 |
| 0 | 1 |

The COMPLEMENT operation may be realized as in Figure 3::

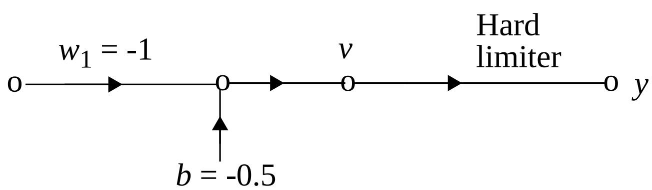


Figure 3: Problem 1.3

The hard limiter input is

$$v = wx + b = -x + 0.5$$

If $x = 1$, then $v = -0.5$, and $y = 0$
If $x = 0$, then $v = 0.5$, and $y = 1$

These conditions agree with truth table 3.

(b) EXCLUSIVE OR operation: Truth table 4

| Inputs | | Output |
|---|---|---|
| $x_1$ | $x_2$ | y |
| 1 | 1 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 0 | 0 | 0 |

This operation is nonlinearly separable, which cannot be solved by the perceptron.

**Problem 1.4**

The Gaussian classifier consists of a single unit with a single weight and zero bias, determined in accordance with Eqs. (1.37) and (1.38) of the textbook, respectively, as follows:

$$\begin{aligned} w &= \frac{1}{\sigma^2}(\mu_1 - \mu_2) \\ &= -20 \end{aligned}$$

$$
\begin{aligned}
b &= \frac{1}{2\sigma^2}(\mu_2^2 - \mu_1^2) \\
&= 0
\end{aligned}
$$

**Problem 1.5**

Using the condition

$$\mathbf{C} = \sigma^2 \mathbf{I}$$

in Eqs. (1.37) and (1.38) of the textbook, we get the following formulas for the weight vector and bias of the Bayes classifier:

$$\mathbf{w} = \frac{1}{\sigma^2}(\mu_1 - \mu_2)$$

$$\mathbf{b} = \frac{1}{2\sigma^2}(\|\mu_1\|^2 - \|\mu_2\|^2)$$