Chapter 1

1. (i) $A^0 = 4, A^1 = 3, A^2 = 2, A^3 = 1$
$A_i = \eta_{ik}A^k \Rightarrow A_0 = 4, A_1 = -3, A_2 = -2, A_3 = -1$
$A_iA^i = 4^2 - 3^2 - 2^2 - 1^2 = 2 > 0 \Rightarrow A^i$ is timelike.

(ii) $x^2 + y^2 = 1 \Rightarrow xdx + ydy = 0 \Rightarrow dx = -\lambda y, dy = \lambda x$
$z = 0 \Rightarrow dz = 0, t = 0 \Rightarrow dt = 0.$
$c^2dt^2 - dx^2 - dy^2 - dz^2 = 0 - \lambda^2x^2 - \lambda^2y^2 - 0 = -\lambda^2 < 0$
Hence the tangent vector is spacelike.

(iii) $\phi \equiv x^2 + y^2 + z^2 - c^2t^2 = 1.$
Normal vector is $(\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}) \equiv (-2c^2t, 2x, 2y, 2z) = A_i$ (say).

$$A^i = -2t, -2x, -2y, -2z$$
$$\Rightarrow A_iA^i = 4c^2t^2 - 4x^2 - 4y^2 - 4z^2 = -4 < 0.$$

Hence the normal vector is spacelike.

(iv) We have $\frac{dx^1}{d\lambda} = r\ \sin\theta$, $\frac{dx^2}{d\lambda} = r\ \cos\theta$, $\frac{dx^3}{d\lambda} = z$, $\frac{dx^0}{d\lambda} = \sqrt{r^2 + z^2}$.

$$n_{ik}\frac{dx^i}{d\lambda}\frac{dx^k}{d\lambda} = -r^2\ \sin^2\theta - r^2\ \cos^2\theta - z^2 + r^2 + z^2 = 0.$$
$$\Rightarrow \text{The vector is null.}$$

2. Using the electromagnetic field tensor F_{ik}, where

$$F_{01} = E_1\ ,\ F_{02} = E_2\ ,\ F_{03} = E_3$$

$$F_{32} = B_1\ , F_{13} = B_2\ , F_{21} = B_3$$

$\mathbf{E} = (E_1, E_2, E_3)$, $\mathbf{B} = (B_1, B_2, B_3)$ being the electric and magnetic field vectors in 3 dimensions, we use the special Lorentz transformation in the form

$$x'^i = L^i{}_k x^k$$

with the non-zero components of $L^i{}_k$ as $L^0{}_0 = \gamma$, $L^0{}_1 = -\gamma v$, $L^1{}_0 = -\gamma v$, $L^1{}_1 = \gamma$, and $L^2{}_2 = L^3{}_3 = 1$.

The tensor transformation law gives $F_{ik} \to F'_{ik}$, where

$$F'_{ik} = L_i{}^m L_k{}^n F_{mn}.$$

We also have $L_0{}^0 = \gamma$, $L_0{}^1 = \gamma v = L_1{}^0$, $L_1{}^1 = \gamma$, $L_2{}^2 = L_3{}^3 = 1$. So, for example

$$\begin{aligned} F'_{01} &= L_0{}^0 L_1{}^1 F_{01} + L_0{}^1 L_1{}^0 F_{10} \\ &= \gamma^2(1 - v^2)F_{01} = F_{01}, \end{aligned}$$

i.e., $E'_1 = E_1$.

It can be easily verified that the other components transform as :

$$E'_2 = \gamma(E_2 - vB_3),\ E'_3 = \gamma(E_3 + vB_2),$$

$$B'_1 = B_1,\ B'_2 = \gamma(B_2 + vE_3),\ B'_3 = \gamma(B_3 - vE_2).$$

3. Let in its rest frame the two components of the length vector of the rod along and perpendicular to the direction of motion be (l_1, l_2). Then

$$\frac{l_2}{l_1} = \tan 60^0 = \sqrt{3}.$$

In the frame S, the component l_2 has the same apparent length l'_2 as before. The length l_1, however, appears contracted to

$$l'_1 = l_1\sqrt{1 - \frac{v^2}{c^2}} = l_1\sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4l_1}{5}.$$

Since $l'_2 = l_2$, we have the apparent angle of inclination of the rod as θ, where

$$\cot\,\theta = \frac{l'_1}{l'_2} = \frac{4l_1/5}{l_2} = \frac{4}{5} \times \frac{1}{\sqrt{3}},$$

i.e., $\theta = \cot^{-1}(4/5\sqrt{3})$.

4. Let the 4-momentum of the photon in the laboratory frame be

$$p^i = (E, E\cos\theta, E\sin\theta, 0)$$

After Lorentz transformation in the rest frame of the mirror it becomes

$$p'^i = [\gamma E(1 + v\cos\theta),\ \gamma E(v + \cos\theta), E\sin\theta, 0].$$

After reflection it will be

$$p'^i_{\rm ref} = [\gamma E(1 + v\cos\theta), -\gamma E(v + \cos\theta), E\sin\theta, 0].$$

Transforming back to the laboratory frame this becomes

$$\begin{aligned} p'^i_{\rm ref} &= [\gamma^2 E\{(1 + v\cos\theta) + v(v + \cos\theta)\}, \gamma^2 E\{-v(1 + v\cos\theta) - (v + \cos\theta)\}, \\ &\quad E\sin\theta, 0] \end{aligned}$$

So $\cos\bar{\theta}$ after reflection will be

$$\begin{aligned} \cos\bar{\theta} &= \frac{|\,p^2\,|}{|\,p^0\,|} = \frac{(1+v^2)\cos\theta + 2v}{1 + 2v\cos\theta + v^2} \\ &= \frac{\cos\theta + \dfrac{2v}{1+v^2}}{1 + \dfrac{2v}{1+v^2}\cos\theta} \\ &= \frac{\cos\theta + \cos\alpha}{1 + \cos\theta\cos\alpha}. \end{aligned}$$

5. The Compton scattering formula in section 1.7.4 tells us that the wavelength change is given by

$$\Delta\lambda = \frac{h}{m_0 c}(1 - \cos\theta)$$

For $\theta = 60^0,\ \cos\theta = \frac{1}{2}$ and $\Delta\lambda = h/2m_0c$.

6. Going back to the definition of F_{ik}, we use the fact that the expressions

$$F_{ik}F^{ik} \text{ and } \epsilon_{ijkl}F^{ij}F^{kl}$$

are invariants. Substituting the values of the components we find that the first is proportional to $B^2 - E^2$ and the second to $\mathbf{B} \cdot \mathbf{E}$.

7. $F_{ik}F^{ik}$ and $\epsilon_{ijkl}F^{ij}F^{kl}$ are invariants and as shown in Q.6, they are $B^2 - E^2$ and $\mathbf{B} \cdot \mathbf{E}$.

Now by a Lorentz transformation we can give arbitrary values to B and E subject to the above invariants.

Consider the Lorentz frame in which B and E are parallel. Then $\mathbf{B} \cdot \mathbf{E} = 0$ gives $BE = 0$. Hence either $B = 0$ or $E = 0$. That is, either the magnetic or the electric field is zero.

8. The equation of motion of the charge is

$$m\frac{du^i}{ds} = qF^i{}_k u^k$$

Let the orbit be in $x^1 - x^2$ plane with the magnetic field in the x^3 -direction.
Then

$$F_{ik} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & -B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The 3-velocity of the charge is $\mathbf{v} = (R\omega \cos\theta, R\omega \sin\theta, 0)$. This corresponds to a 4-velocity

$$u^i = \frac{1}{\sqrt{1 - R^2\omega^2}}(1, R\omega \cos\theta, R\omega \sin\theta, 0)$$

The equation of motion then has these components:

$$\frac{m}{\sqrt{1 - R^2\omega^2}}\frac{d}{dt}\left(\frac{R\omega \cos\theta}{\sqrt{1 - R^2\omega^2}}\right) = -qB\frac{R\omega \sin\theta}{\sqrt{1 - R^2\omega^2}}$$

$$\frac{m}{\sqrt{1 - R^2\omega^2}}\frac{d}{dt}\left(\frac{R\omega \sin\theta}{\sqrt{1 - R^2\omega^2}}\right) = qB\frac{R\omega \cos\theta}{\sqrt{1 - R^2\omega^2}}$$

Both these equations lead to

$$\frac{mR\omega}{1-R^2\omega^2}\dot{\theta} = qB\frac{R\omega}{\sqrt{1-R^2\omega^2}},$$

Since $\dot{\theta} = \omega$,

$$B = \frac{m\omega}{q\sqrt{1-R^2\omega^2}}.$$

9. The Doppler spectral shift is given by

$$1 + z = \frac{1 + v\cos\theta}{\sqrt{1-v^2}}.$$

For zero shift $z = 0$ and

$$\sqrt{1-v^2} = 1 + v\cos\theta,$$

$$\text{i.e., } \theta = \cos^{-1}\frac{\sqrt{1-v^2}-1}{v}.$$

10. We need to estimate v, the orbital velocity of the Earth. The Earth–Sun distance is $\approx$ 150 million km $\approx 1.5 \times 10^{13}$ cm. The circumference of the orbit is (assuming circular shape)

$$l = 2\pi \times 1.5 \times 10^{13}\text{cm}.$$

This distance is covered by the Earth in time $T = 1$ year.

$$T \cong 365 \times 24 \times 3600 \text{ second } \approx 3.15 \times 10^7 s$$

Therefore

$$v = \frac{2\pi \times 1.5 \times 10^{13}}{3.15 \times 10^7} \cong 3 \times 10^6 \text{cm s}^{-1}$$

The order of magnitude of the effect, assuming $\sin\theta = 0(1)$, is

$$\alpha \cong \frac{2v}{c} \approx \frac{6 \times 10^6}{3 \times 10^{10}} \text{ radians} = 2 \times 10^{-4} \text{ radians} \approx 40 \text{ arsec}.$$

11. The answer is "no". For, if it were "yes", in the rest frame of the electron its 4-momentum vector would be $(m_o c^2, 0, 0, 0)$. This must equal the sum of 4-momenta of decay products. If the energy of the emitted photon were $h\nu$ and the electron, after decay, had a γ-factor ≥ 1, we would have

$$m_o c^2 = \gamma m_o c^2 + h\nu.$$

Since the right hand circle exceeds the left hand side for $\gamma \geq 1$, $\nu \geq 0$, we get a contradiction.

12. The 4-momentum of the original particle is $(M_0, 0, 0,)$. The decay products have γ- factors, γ, γ_2, γ_3 releated by

$$\gamma_2 M_2 = \gamma_3 M_3 = X(\text{say}).$$

Choose the x-axis along the direction of M_2 and y-axis along the direction of M_3. Then the 4-momenta of M_2, M_3 are $(\gamma_2 M_2, \gamma_2 v_2 M_2, 0, 0)$ and $(\gamma_3 M_3, 0, \gamma_3 v_3 M_3, 0)$, respectively. The third decay product will move at an angle $\pi + \alpha$ with the x- axis, say, with velocity v_1. Then its 4-momentum is $(\gamma_1 M_1, -\gamma_1 M_1 v_1 \cos\alpha, -\gamma_1 M_1 v_1 \sin\alpha, 0)$. The conservation of momentum then gives these three relations:

$$M_0 = \gamma_1 M_1 + \gamma_2 M_2 + \gamma_3 M_3 = \gamma_1 M_1 + 2X \text{ where } X = \text{ energy } of M_2 \text{ and } M_3.$$

$$0 = -\gamma_1 M_1 v_1 \;\cos\alpha + \gamma_2 M_2 v_2$$

$$0 = -\gamma M_1 v_1 \;\sin\alpha + \gamma_3 M_3 v_3.$$

We also have the identity $\gamma^2 - \gamma^2 v^2 = 1$ for all γs.
Thus $\gamma_2^2 M_2^2 - \gamma_2^2 M_2^2 v_2^2 = M_2^2 \Rightarrow X^2 = M_2^2 + X^2 v_2^2$.
From the second and third momentum equations we get

$$\gamma_1^2 M_1^2 v_1^2 = \gamma_2^2 M_2^2 v_2^2 + \gamma_3^2 M_3^2 v_3^2 = 2X^2 - (M_2^2 + M_3^2),$$

$$\text{i.e., } M_1^2(\gamma_1^2 - 1) = 2X^2 - (M_2^2 + M_3^2),$$