CHAPTER 2

Solution Manual

Exact

2.1. (a) $d\Omega = \sin\theta \, d\theta \, d\phi$

$$\Omega_A = \int_{45^\circ}^{60^\circ}\int_{30^\circ}^{60^\circ} d\Omega = \int_{\pi/4}^{\pi/3}\int_{\pi/6}^{\pi/3} \sin\theta \, d\theta \, d\phi$$

$$= (\phi)\Big|_{\pi/4}^{\pi/3}(-\cos\theta)\Big|_{\pi/6}^{\pi/3}$$

$$= \left(\frac{\pi}{3} - \frac{\pi}{4}\right)(-0.5 + 0.866)$$

$$\Omega_A = \left(\frac{\pi}{12}\right)(0.366) = 0.09582 \text{ sterads}$$

$$\Omega_A = \begin{cases} 0.09582 \text{ sterads} \\ 0.09582\left(\dfrac{180}{\pi}\right)\left(\dfrac{180}{\pi}\right) = 314.5585 \ (\text{degrees})^2 \end{cases}$$

Approximate

$$\Omega_A \simeq \left(\frac{\pi}{3} - \frac{\pi}{4}\right)\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

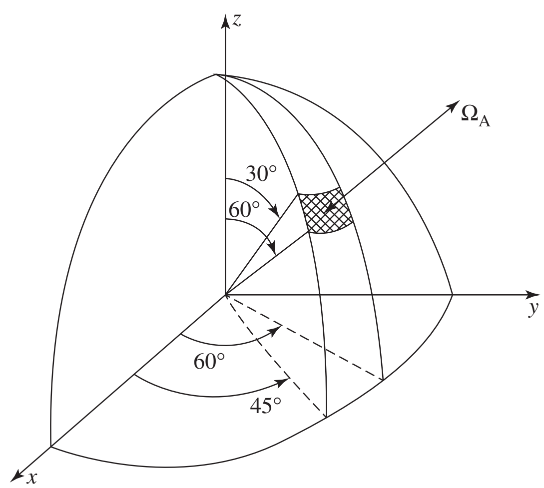
$$\simeq \left(\frac{\pi}{12}\right)\left(\frac{\pi}{6}\right) = \frac{\pi^2}{72}$$

$\Omega_A \simeq 0.13708$ sterads

$\Omega_A \simeq (60 - 45)(60 - 30)$

$\simeq 450 \ (\text{degrees})^2$ or error of

$$\left(\frac{450 - 314.5585}{314.5585}\right) \times 100 = 43.06\%$$



Antenna Theory: Analysis and Design, Fourth Edition. Constantine A. Balanis.

Companion Website: www.wiley.com/go/antennatheory4e

(b) $D_0 = \dfrac{4\pi}{\Omega_A(\text{sterads})} = \dfrac{4\pi}{0.09582} = 131.1456 \text{ (dimensionless)}$

$= 10\log_{10}(131.1456) = 21.1775 \text{ dB}$

or

$$D_0 = \frac{4\pi\left(\frac{180}{\pi}\right)\left(\frac{180}{\pi}\right)}{\Omega_A(\text{degrees})^2} = 131.1456 \text{ (dimensionless)} = 21.1775 \text{ dB}$$

$$D_0 = \begin{cases} 131.1456 \text{ (dimensionless)} \\ 21.1775 \text{ (dB)} \end{cases}$$

2.2. $\underline{\mathcal{W}} = \underline{\mathcal{E}} \times \underline{\mathcal{H}} = \text{Re}\left[\underline{E}e^{j\omega t}\right] \times \text{Re}\left[\underline{H}e^{j\omega t}\right]$

Using the identity $\text{Re}\left[\underline{A}e^{j\omega t}\right] = \frac{1}{2}\left[\underline{E}e^{j\omega t} + \underline{E}^* e^{-j\omega t}\right]$

The instant Poynting vector can be written as

$$\underline{\mathcal{W}} = \left\{\frac{1}{2}[\underline{E}e^{j\omega t} + \underline{E}^* e^{-j\omega t}]\right\} \times \left\{\frac{1}{2}[\underline{H}e^{j\omega t} + \underline{H}^* e^{-j\omega t}]\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{2}[\underline{E}\times\underline{H}^* + \underline{E}^* \times \underline{H}] + \frac{1}{2}[\underline{E}\times\underline{H}e^{j2\omega t} + \underline{E}^* \times \underline{H}^* e^{-j\omega t}]\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{2}[\underline{E}\times\underline{H}^* + (\underline{E}\times\underline{H}^*)^*] + \frac{1}{2}[\underline{E}\times\underline{H}e^{j2\omega t} + (\underline{E}\times\underline{H}e^{j\omega t})^*]\right\}$$

Using the above identity again, but this time in reverse order, we can write that

$$\underline{\mathcal{W}} = \frac{1}{2}[\text{Re}(\underline{E}\times\underline{H}^*)] + \frac{1}{2}[\text{Re}(\underline{E}\times\underline{H}e^{j2\omega t})]$$

2.3. (a) $\underline{W}_{\text{rad}} = \dfrac{1}{2}\text{Re}[\underline{E}\times\underline{H}^*] = \dfrac{E^2}{2\eta}\hat{a}_r = \dfrac{5^2}{2(120\pi)}\hat{a}_r = 0.03315\hat{a}_r \text{ Watts/m}^2$

(b) $P_{\text{rad}} = \oint_s W_{\text{rad}}\, ds = \int_0^{2\pi}\int_0^{\pi} (0.03315)(r^2 \sin\theta\, d\theta\, d\phi)$

$= \int_0^{2\pi}\int_0^{\pi} (0.03315)(100)^2 \sin\ \theta\, d\theta\, d\phi$

$= 2\pi(0.03315)(100)^2 \int_0^{\pi} \sin\theta\, d\theta = 2\pi(0.03315)(100)^2 \cdot (2)$

$= 4165.75 \text{ Watts}$

2.4. (a) $U(\theta) = \cos\ \theta$

$U(\theta_h) = 0.5 = \cos\ \theta_h \Rightarrow \theta_h = \cos^{-1}(0.5) = 60°$

$\Rightarrow \Theta_h = 2(60°) = 120° = \dfrac{2\pi}{3}$ rads.

$U(\theta_n) = 0 = \cos\ \theta_n \Rightarrow \theta_n = \cos^{-1}(0) = 90°$

$\Rightarrow \Theta_n = 2(90°) = 180° = \pi$ rads.

(b) $U(\theta) = \cos^2\theta$

$$U(\theta_h) = 0.5 = \cos^2\theta_h \Rightarrow \theta_h = \cos^{-1}(0.5)^{1/2} = 45^\circ$$
$$\Rightarrow \Theta_h = 2(45) = 90^\circ = \pi/2 \text{ rads}$$
$$U(\theta_n) = 0 = \cos^2\theta_n \Rightarrow \theta_n = \cos^{-1}(0) = 90^\circ$$
$$\Rightarrow \Theta_n = 2(90^\circ) = 180^\circ = \pi \text{ rads}$$

(c) $U(\theta) = \cos(2\theta)$

$$U(\theta_h) = 0.5 = \cos(2\theta_h) \Rightarrow \theta_h = \frac{1}{2}\cos^{-1}(0.5) = 30^\circ$$
$$\Rightarrow \Theta_h = 2(30^\circ) = 60^\circ = \pi/3 \text{ rads}$$
$$U(\theta_n) = 0 = \cos(2\theta_n) \Rightarrow \theta_n = \frac{1}{2}\cos^{-1}(0) = 45^\circ$$
$$\Rightarrow \Theta_n = 2(45^\circ) = 90^\circ = \pi/2 \text{ rads}$$

(d) $U(\theta) = \cos^2(2\theta)$

$$U(\theta_h) = 0.5 = \cos^2(2\theta_h) \Rightarrow \theta_h = \frac{1}{2}\cos^{-1}(0.5)^{1/2} = 22.5^\circ$$
$$\Rightarrow \Theta_h = 2(22.5^\circ) = 45^\circ = \frac{\pi}{4} \text{ rads}$$
$$U(\theta_n) = 0 = \cos^2(2\theta_n) \Rightarrow \theta_n = \frac{1}{2}\cos^{-1}(0) = 45^\circ$$
$$\Rightarrow \Theta_n = 2(45^\circ) = 90^\circ = \pi/2 \text{ rads}$$

(e) $U(\theta) = \cos(3\theta)$

$$U(\theta_h) = \cos(3\theta_h) = 0.5 \Rightarrow \theta_h = \frac{1}{3}\cos^{-1}(0.5) = 20^\circ$$
$$\Rightarrow \Theta_h = 2(20^\circ) = 40^\circ = 0.698 \text{ rads}$$
$$U(\theta_n) = \cos(3\theta_n) = 0 \Rightarrow \theta_n = \frac{1}{3}\cos^{-1}(0) = 30^\circ$$
$$\Rightarrow \Theta_n = 2(30^\circ) = 60^\circ = \pi/3 \text{ rads}$$

(f) $U(\theta) = \cos^2(3\theta)$

$$U(\theta_h) = 0.5 = \cos^2(3\theta_h) \Rightarrow \theta_h = \frac{1}{3}\cos^{-1}(0.5)^{1/2} = 15^\circ$$
$$\Rightarrow \Theta_h = 2(15^\circ) = 30^\circ = \pi/6 \text{ rads}$$
$$U(\theta_n) = 0 = \cos^2(3\theta_n) \Rightarrow \theta_n = \frac{1}{3}\cos^{-1}(0) = 30^\circ$$
$$\Rightarrow \Theta_n = 2(30^\circ) = 60^\circ = \pi/3 \text{ rads}$$

2.5. Using the results of Problem 2.4 and a nonlinear solver to find the half power beamwidth of the radiation intensity represented by the transcentendal functions, we have that:

(a) $U(\theta) = \cos\theta\ \cos(2\theta) \Rightarrow \begin{cases} \text{HPBW} = 55.584^\circ \\ \text{FNBW} = 90^\circ \end{cases}$

(b) $U(\theta) = \cos^2\theta \ \cos^2(2\theta) \Rightarrow \begin{cases} \text{HPBW} = 40.985^\circ \\ \text{FNBW} = 90^\circ \end{cases}$

(c) $U = \cos\theta \ \cos(3\theta) \Rightarrow \begin{cases} \text{HPBW} = 38.668^\circ \\ \text{FNBW} = 60^\circ \end{cases}$

(d) $U = \cos^2\theta \ \cos^2(3\theta) \Rightarrow \begin{cases} \text{HPBW} = 28.745^\circ \\ \text{FNBW} = 60^\circ \end{cases}$

(e) $U = \cos(2\theta) \ \cos(3\theta) \Rightarrow \begin{cases} \text{HPBW} = 34.942^\circ \\ \text{FNBW} = 60^\circ \end{cases}$

(f) $U = \cos^2(2\theta) \ \cos^2(3\theta) \Rightarrow \begin{cases} \text{HPBW} = 25.583^\circ \\ \text{FNBW} = 60^\circ \end{cases}$

2.6. (a) $D_0 = \dfrac{4\pi U_{max}}{P_{rad}} = \dfrac{4\pi(200\times10^{-3})}{0.9(125.66\times10^{-3})} = 22.22 = 13.47 \text{ dB}$

$G_0 = \varepsilon_{cd} D_0 = 0.9(22.22) = 20 = 13.01 \text{ dB}$

(b) $D_0 = \dfrac{4\pi U_{max}}{P_{rad}} = \dfrac{4\pi(200\times10^{-3})}{(125.66\times10^{-3})} = 20 = 13.01 \text{ dB}$

$G_0 = \varepsilon_{cd} D_0 = 0.9\cdot(20) = 18 = 12.55 \text{ dB}$

2.7. $\boxed{U = B_0 \ \cos^2\theta}$

(a)
$$P_{rad} = \int_0^{2\pi}\int_0^{\pi/2} U \ \sin\theta \, d\theta = 2\pi B_0 \int_0^{\pi/2} \cos^2\theta \ \sin\theta \, d\theta$$
$$= 2\pi B_0 \int_0^{\pi/2} \cos^2 \ \theta \ d\,(-\cos\theta)$$
$$P_{rad} = -2\pi B_0 \frac{\cos^3\theta}{3}\bigg|_0^{\pi/2} = -2\pi B_0\left[\frac{-1}{3}\right] = \frac{2\pi}{3}B_0 = 10 \Rightarrow B_0 = \frac{15}{\pi}$$
$$U = \frac{15}{\pi}\cos^2\theta \ \Rightarrow W_{rad}\bigg|_{max} = \frac{U}{r^2}\bigg|_{max} = \frac{15}{\pi}\frac{\cos^2\theta}{r^2}\bigg|_{max}$$
$$= \frac{15}{\pi(10^3)^2} = 4.7746\times10^{-6} \text{ Watts/m}^2 @ \ \theta = 0^\circ$$
$$W_{rad}\bigg|_{max} = 4.7746\times10^{-6} \text{ Watts/m}^2 @ \ \theta = 0^\circ$$

(b)
$$\Omega_A \text{ (exact)} = \int_0^{2\pi}\int_0^{\pi} U_n \ \cos^2\theta \ \sin\theta \, d\theta \, d\phi$$
$$\Omega_A \text{ (exact)} = \frac{2\pi}{3} \text{ steradians} = 2.0944 \text{ sterads} = 6,875.51 \text{ (degrees)}^2$$
$$U = 0.5 = \cos^2\theta_h \Rightarrow \theta_h = \cos^{-1}(0.5)^{1/2} = 45^\circ$$
$$\Rightarrow \Theta_h = 2(45^\circ) = 90^\circ = \pi/2 \text{ rads}$$
$$\Omega_A\left(\begin{matrix}\text{Kraus'}\\ \text{approx}\end{matrix}\right) = \Theta_h^2 = (\pi/2)^2 = \frac{\pi^2}{4} = 2.4674 \text{ sterads} = 8,099.997 \text{ (degrees)}^2$$

(c) $$D_0\,(\text{exact}) = \frac{4\pi}{\Omega_A(\text{exact})} = \frac{4\pi}{2\pi/3} = 6 = 7.782\text{ dB}$$

$$D_0\,(\text{approx/Kraus'}) = \frac{4\pi}{\Omega_A(\text{approx})} = \frac{4\pi}{\pi^2/4} = \frac{16}{\pi} = 5.093 = 7.0697$$

(d) G_0 Assuming lossless antenna ($P_{in} = P_{\text{rad}}$)

$$G_0\,(\text{exact}) = D_0\,(\text{exact}) = 6 = 7.782\text{ dB}$$

$$G_0\,(\text{approx}) = D_0\,(\text{approx}) = 5.093 = 7.0697\text{ dB}$$

$$\boxed{U = B_0 \cos^3\theta}$$

(a) $$P_{\text{rad}} = -2\pi B_0\left(-\frac{1}{4}\right) = \frac{\pi}{2}B_0 = 10 \Rightarrow B_0 = 20/\pi$$

$$W_{\text{rad}}\Big|_{\text{max}} = \frac{20}{\pi}\frac{1}{\pi^2} = \frac{20}{\pi}\times 10^{-6} = 6.366\times 10^{-6}\text{ Watts/m}^2$$

(b) Ω_A (exact) $= (\pi/2) = 1.5708$ sterads

$$U = 0.5 = \cos^3\theta_h \Rightarrow \theta_h \ \cos^{-1}(0.5)^{1/3} = 37.467^\circ$$

$$\Rightarrow \Theta_h = 2(37.467^\circ) = 74.934^\circ = 1.30785\text{ rads}$$

$$\Omega_A\,(\text{approx}) = (1.30785)^2 = 1.71\text{ sterads}$$

(c) D_0 (exact) $= 4\pi/(\pi/2) = 8 = 9.031$ dB

$$D_0\,(\text{approx}) = \frac{4\pi}{1.71} = 7.347 = 8.66\text{ dB}$$

(d) Assuming lossless antenna ⇒ Gain = Directivity (see part c)

2.8. $\underline{E}_a = \hat{a}_\theta E_a \sin^{1.5}\theta \frac{e^{-jkr}}{r} \Rightarrow U_n = (\sin^{1.5}\theta)^2 = \sin^3\theta$ **Normalized U_n**

(a) $$D_0 = \frac{4\pi U_{\text{max}}}{P_{\text{rad}}},\ U_{\text{max}} = U_n|_{\text{max}} = \sin^3\theta\Big|_{\substack{\text{max}\\ \theta=\theta\,\text{max}}} = 1,\ \theta_{\text{max}} = 90^\circ$$

$$P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi} U_n \sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\int_0^{\pi} \sin^3\theta\sin\theta\, d\theta\, d\phi = 2\pi\int_0^{\pi}\sin^4\theta\, d\theta$$

$$= 2\pi\left[-\frac{\sin^3\theta\cos\theta}{4}\Big|_0^{\pi} + \frac{3}{4}\int_0^{\pi}\sin^2\theta\, d\theta\right] = 2\pi\left[\frac{3}{4}\left(\frac{1}{2}-\frac{1}{4}\sin(2\theta)\right)_0^{\pi}\right]$$

$$= 2\pi\left[\frac{3}{4}\left(\frac{\pi}{2}\right)\right] = \frac{3\pi^2}{4}$$

$$D_0 = \frac{4\pi U_{\text{max}}}{P_{\text{rad}}} = \frac{4\pi(1)}{3\pi^2/4} = \frac{16}{3\pi} = \boxed{1.698 = 2.298\text{ dB}}$$

(b) $U_n = \sin^3\theta, U_{n\text{max}} = 1, \theta_{\text{max}} = 90^\circ$

$$U_n\Big|_{\theta=\theta_h} = 0.5 = \sin^3\theta_h \Rightarrow \theta_h = \left[\sin^{-1}(0.5^4)^3\right] = \sin^{-1}(0.794) = 52.533^\circ$$

$$\text{HPBW} = \Theta_h = 2(\theta_{\text{max}} - \theta_h) = 2(90 - 52.533)$$

$$\boxed{\Theta_h = 2(37.467) = 74.934^\circ}$$

P1: OTE/SPH P2: OTE
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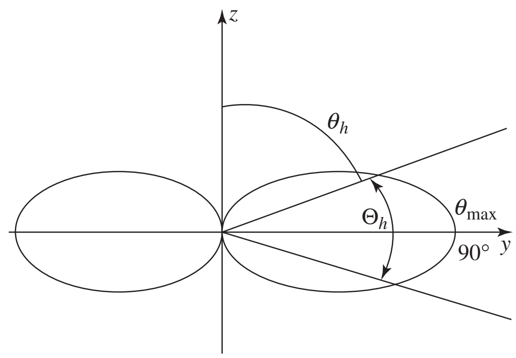

(c) Because pattern is **omnidirectional**:

$$D_0(\text{McDonald}) = \frac{101}{\text{HPBW} - 0.0027(\text{HPBW})^2} = \frac{101}{74.934 - 0.0027(74.934)^2}$$

$$D_0(\text{McDonald}) = \frac{101}{74.934 - 15.161} = \frac{101}{59.773} = \boxed{1.690 = 10\log_{10}(1.690) = 2.278}$$

(d) Because pattern is **omnidirectional**:

$$D_0(\text{Pozar}) = -172.4 + 191\sqrt{0.818 + \frac{1}{\text{HPBW}}} = -172.4 + 191\sqrt{0.818 + \frac{1}{74.934}}$$

$$= -172.4 + 191(0.912) = -172.4 + 174.150 = 1.750$$

$$P_0(\text{Pozar}) = \boxed{1.750 = 10\log_{10}(1.750) = 2.431 \text{ dB}}$$

(e) Computer Program Directivity: $\boxed{D_0 = 1.693 = 2.2864 \text{ dB}}$

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Input parameters:
-----------------
The lower bound of theta in degrees = 0
The upper bound of theta in degrees = 180
The lower bound of phi in degrees = 0
The upper bound of phi in degrees = 360

Output parameters:
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Radiated power (watts) = 7.4228
Directivity (dimensionless) = 1.6930
Directivity (dB) = 2.2864
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2.9. $U(\theta, \phi) = \cos^n(\theta) \qquad 0 \le \theta \le \pi/2, 0 \le \phi \le 2\pi$

(a) $U_n(\theta_n, \phi) = 0.5 = \cos^n(5°) = [\cos(5°)]^n = (0.99619)^n$

$$0.5 = (0.99619)^n$$

$$\log_{10}(0.5) = \log[(0.99619)^n] = n\log_{10}(0.99619) = n(-0.00166)$$

$$-0.30103 = -0.00166n$$

$$\boxed{n = 181.34}$$

(b) $U(\theta,\phi) = \cos^{181.34}(\theta);\ U_{\max} = 1, \theta = 0°$

$$P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi/2} U(\theta,\phi)\sin\theta\, d\theta\, d\phi = 2\pi\int_0^{\pi/2}\cos^{181.34}(\theta)\sin\theta\, d\theta$$

$$= 2\pi\left[-\frac{\cos^{182.34}(\theta)}{182.34}\right]_0^{\pi/2} = \left[-0 + \frac{1}{182.34}\right]2\pi = \frac{2\pi}{182.34} = 0.03446$$

$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}} = \frac{4\pi(1)}{2\pi}(182.34) = 2(182.34) = 364.68$$

$$\boxed{D_0 = 364.68 = 25.62 \text{ dB}}$$

(c) *Kraus' Approximation* (2.27):

$$D_0 \simeq \frac{41,253}{\Theta_{1d}\Theta_{2d}} = \frac{41,253}{(10)(10)} = 412.53 = 26.15 \text{ dB}$$

$$\boxed{D_0 \simeq 412.53 = 26.15 \text{ dB}}$$

(d) *Tai & Pereira* (2.30b):

$$D_0 \simeq \frac{72{,}815}{\Theta_{1d}^2 + \Theta_{2d}^2} = \frac{72{,}815}{2(10)^2} = \frac{72{,}815}{200} = 364.075 = 25.61 \text{ dB}$$

$$\boxed{D_0 \simeq 364.075 = 25.61 \text{ dB}}$$

2.10.

$$U(\theta,\phi) = \left\{\begin{matrix} 1 & 0° \le \theta \le 20° \\ 0.342\csc(\theta) & 20° \le \theta \le 60° \\ 0 & 60° \le \theta \le 180° \end{matrix}\right\} 0° \le \phi \le 360°$$

$$P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta\, d\theta\, d\phi$$

$$= 2\pi\left[\int_0^{20°}\sin\theta\, d\theta + \int_{20°}^{60°} 0.342\csc(\theta)\times\sin\theta\, d\theta\right]$$

$$= 2\pi\left\{-\cos\theta\Big|_0^{\pi/9} + 0.342\cdot\theta\Big|_{\pi/9}^{\pi/3}\right\}$$

$$= 2\pi\left\{\left[-\cos\left(\frac{\pi}{9}\right)+1\right] + 0.342\left(\frac{\pi}{3}-\frac{\pi}{9}\right)\right\}$$

$$= 2\pi\left\{[-0.93969+1] + 0.342\pi\left(\frac{2}{9}\right)\right\}$$

$$= 2\pi\{0.06031 + 0.23876\} = 1.87912$$

$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}} = \frac{4\pi(1)}{1.87912} = 6.68737 = 8.25255 \text{ dB}$$

2.11. (a) $D_0 \simeq \dfrac{41{,}253}{\Theta_{1d}\Theta_{2d}} = \dfrac{41{,}253}{30(35)} = 39.29 = 15.94 \text{ dB}$

$$A_{em} = \frac{\lambda^2}{4\pi}D_0$$

(b) $D_0 \simeq \dfrac{72{,}815}{\Theta_{1d}^2 + \Theta_{2d}^2} = \dfrac{72{,}815}{(30)^2 + (35)^2} = 34.27 = 15.35 \text{ dB}$

$$A_{em} = \frac{\lambda^2}{4\pi}D_0$$

2.12. $D_0 = \dfrac{4\pi U_{\max}}{P_{\text{rad}}}$

(a) $U = \sin\theta \sin\phi$ for $0 \le \theta \le \pi, 0 \le \phi \le \pi$

$U\big|_{\max} = 1$ and it occurs when $\theta = \phi = \pi/2$.

$$P_{\text{rad}} = \int_0^\pi \int_0^\pi U \sin\theta \, d\theta \, d\phi = \int_0^\pi \sin\phi \, d\phi \int_0^\pi \sin^2\theta \, d\theta = 2\left(\frac{\pi}{2}\right) = \pi.$$

Thus $D_0 = \dfrac{4\pi(1)}{\pi} = 4 = 6.02$ dB

The half-power beamwidths are equal to

$$\text{HPBW (az.)} = 2[90° - \sin^{-1}(1/2)] = 2(90° - 30°) = 120°$$
$$\text{HPBW (el.)} = 2[90° - \sin^{-1}(1/2)] = 2(90° - 30°) = 120°$$

In a similar manner, it can be shown that for the following:

(b) $U = \sin\theta \sin^2\phi \Rightarrow D_0 = 5.09 = 7.07$ dB

$$\text{HPBW (el.)} = 120°, \text{HPBW (az.)} = 90°$$

(c) $U = \sin\theta \sin^3\phi \Rightarrow D_0 = 6 = 7.78$ dB

$$\text{HPBW (el.)} = 120°, \text{HPBW (az.)} = 74.93°$$

(d) $U = \sin^2\theta \sin\phi \Rightarrow D_0 = 12\pi/8 = 4.71 = 6.73$ dB

$$\text{HPBW (el.)} = 90°, \text{HPBW (az.)} = 120°$$

(e) $U = \sin^2\theta \sin^2\phi \Rightarrow D_0 = 6 = 7.78$ dB

$$\text{HPBW (az.)} = \text{HPBW (el.)} = 90°$$

(f) $U = \sin^2\theta \sin^3\phi \Rightarrow D_0 = 9\pi/4 = 7.07 = 8.49$ dB

$$\text{HPBW (el.)} = 90°, \text{HPBW (az.)} = 74.93°$$

2.13. $U = \sin\theta \cos^2\phi$, $0 \le \theta \le 180°$, $90° \le \phi \le 270°$

(a) $D_0 = \dfrac{4\pi U_{\max}}{P_{\text{rad}}}$, $U_{\max} = \sin\theta \cos^2\phi\Big|_{\substack{\theta=90° \\ \phi=180°}} = 1$

$$P_{\text{rad}} = \int_{\pi/2}^{3\pi/2}\int_0^{\pi} U(\theta,\phi)\sin\theta\, d\theta\, d\phi = \int_{\pi/2}^{3\pi/2}\int_0^{\pi} \sin\theta\cos^2\phi\sin\theta\, d\theta\, d\phi$$

$$= \int_{\pi/2}^{3\pi/2} \cos^2\phi\, d\phi \int_0^{\pi} \sin^2\theta\, d\theta$$

$$P_{\text{rad}} = \left(\frac{\pi}{2}\right)\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}$$

$$D_0 = \frac{4\pi(1)}{\pi^2/4} = \frac{16}{\pi} = 5.09296 = 10\log_{10}(5.09296) = 7.0697 \text{ dB}$$

$$\boxed{D_0(\text{exact}) = 5.09296(\text{dim}) = 7.0697 \text{ dB}}$$

(b) Azimuth (Horizontal) Principal Plane ($\theta = 90°$):

$$U(\theta = 90°) = \sin\theta\cos^2\phi|_{\theta=90°} = \cos^2\phi$$

$$U_h = \cos^2\phi|_{\phi=\phi_h} = 0.5 \Rightarrow \phi_h = \cos^{-1}(\pm\sqrt{0.5}) = \cos^{-1}(\pm 0.707) = 135°$$

$$\Phi_h(az) = 2(180 - 135) = 2(45°) = 90°$$

$$\boxed{\Phi_h(az) = 90°}$$

(c) Elevation (vertical) Principal plane ($\phi = 180°$):

$$U(\phi = 180°) = \sin\theta\cos^2\phi|_{\phi=180°} = \sin\theta$$

$$U_h = \sin\theta|_{\theta=\theta_h} = 0.5 \Rightarrow \theta_h = \sin^{-1}(0.5) = 30°$$

$$\Theta_h = 2(90° - 30°) = 2(60) = 120°$$

$$\boxed{\Theta_h\ (\text{elev}) = 120°}$$

(d) **Either:** $D_0(\text{Kraus}) = \dfrac{41{,}253}{\Phi_h\Theta_h} = \dfrac{41{,}253}{90°(120°)} = 3.8197 = 5.82 \text{ dB}$

$$\boxed{D_0(\text{Kraus}) = 3.8197 \text{ dim} = 5.82 \text{ dB}}$$

or:

$$D_0(\text{Tai \& Pereira}) = \frac{72{,}815}{\Phi_h^2 + \Theta_h^2} = \frac{72{,}815}{(90°)^2 + (120°)^2} = \frac{72{,}815}{25{,}500} = 3.236$$

$$\boxed{D_0(\text{T\&P}) = 3.236 \text{ dim} = 5.1 \text{ dB}}$$

2.14. Using the half-power beamwidths found in Problem 2.12, the directivity for each intensity using Kraus' and Tai & Pereira's formulas is given by
$U = \sin\theta\sin\phi$;

(a) $D_0 \simeq \dfrac{41253}{\Theta_{1d}\Theta_{2d}} = \dfrac{41253}{120(120)} = 2.86 = 4.57 \text{ dB}$

(b) $D_0 \simeq \dfrac{72{,}815}{\Theta_{1d}^2 + \Theta_{2d}^2} = \dfrac{72{,}815}{(120)^2 + (120)^2} = 2.53 = 4.03$ dB

$U = \sin\theta \sin^2\phi$;
(a) $D_0 \simeq 3.82 = 5.82$ dB
(b) $D_0 \simeq 3.24 = 5.10$ dB
$U = \sin\theta \sin^3\phi$;
(a) $D_0 \simeq 4.59 = 6.62$ dB
(b) $D_0 \simeq 3.64 = 5.61$ dB
$U = \sin^2\theta \sin\phi$;
(a) $D_0 \simeq 3.82 = 5.82$ dB
(b) $D_0 \simeq 3.24 = 5.10$ dB
$U = \sin^2\theta \sin^2\phi$;
(a) $D_0 \simeq 5.09 = 7.07$ dB
(b) $D_0 \simeq 4.49 = 6.53$ dB
$U = \sin^2\theta \sin^3\phi$;
(a) $D_0 \simeq 6.12 = 7.87$ dB
(b) $D_0 \simeq 5.31 = 7.25$ dB

2.15. (a) $D_0 = \dfrac{4\pi}{\Theta_{1r}\Theta_{2r}} = \dfrac{4\pi}{(1.5064)^2} = 5.5377 = 7.433$ dB

(b) $D_0 = \dfrac{32\ln(2)}{\Theta_{1r}^2 + \Theta_{2r}^2} = \dfrac{32\ln(2)}{(1.5064)^2 + (1.5064)^2} = 4.88725 = 6.8906$ dB

2.16. (a) $D_0 = \dfrac{4\pi U_{\max}}{P_{\text{rad}}} = \dfrac{U_{\max}}{U_0}$

$$P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi} U \sin\theta \, d\theta \, d\phi = 2\pi \int_0^{\pi} U \ \sin\theta \, d\theta = 2\pi \left\{ \int_0^{30^\circ} \sin\theta \, d\theta + \int_{30^\circ}^{60^\circ} (0.5) \ \sin\theta \, d\theta + \int_{60^\circ}^{90^\circ} (0.1) \ \sin\theta \, d\theta \right\}$$

$$= 2\pi \left\{ (-\cos\theta)\Big|_0^{30^\circ} + \left(-\frac{\cos\theta}{2}\right)\Big|_{30^\circ}^{60^\circ} + (-0.1\cos\theta)\Big|_{60^\circ}^{90^\circ} \right\}$$

$$= 2\pi \left\{ (-0.866 + 1) + \left(\frac{-0.5 + 0.866}{2}\right) + \left(\frac{-0 + 0.5}{10}\right) \right\}$$

$$P_{\text{rad}} = 2\pi\{-0.866 + 1 - 0.25 + 0.433 + 0.05\} = 2\pi(0.367)$$

$$= 0.734\pi = 2.3059$$

$$D_0 = \frac{1(4\pi)}{2.3059} = 5.4496 = 7.3636 \text{ dB}$$

(b) $D_0(\text{dipole}) = 1.5 = 1.761$ dB
$D_0(\text{above dipole}) = (7.3636 - 1.761)$ dB $= 5.6026$ dB
$D_0(\text{above dipole}) = \frac{5.45}{1.5} = 3.633 = 5.603$ dB

2.17. (a) $$P_{rad} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\sin^2\phi\, d\phi \int_0^{\pi/2}\cos^4\theta\ \sin\theta\, d\theta$$

$$= (\pi)\left(\frac{1}{5}\right) = \frac{\pi}{5}$$

$$U_{max} = U(\theta = 0°,\ \phi = \pi/2) = 1$$

$$D_0 = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi}{(\pi/5)} = 20 = 13.0 \text{ dB}$$

(b) Elevation Plane: θ varies, ϕ fixed

$\Rightarrow$ Choose $\phi = \pi/2$.

$U(\theta, \phi = \pi/2) = \cos^4\theta, \quad 0 \le \theta \le \pi/2.$

$$\cos^4\left[\frac{\text{HPBW(el.)}}{2}\right] = \frac{1}{2}$$

$$\text{HPBW(el.)} = 2\cos^{-1}\{\sqrt{0.5}\}^{1/2} = 65.5°$$

2.18. (a) $$P_{rad} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta\, d\theta\, d\phi = 2\pi$$

$$\cdot\left\{\int_0^{30°}\sin\theta\, d\theta + \int_{30°}^{90°}\frac{\cos\theta\sin\theta}{0.866}d\theta\right\}$$

$$= 2\pi\left\{\int_0^{\pi/6}\sin\theta\, d\theta + \int_{\pi/6}^{\pi/2}\frac{1}{0.866}\cos\theta\sin\theta\, d\theta\right\}$$

$$= 2\pi\left\{-\cos\theta|_0^{\pi/6} + \frac{1}{0.866}\left(-\frac{\cos^2\theta}{2}\right)\Big|_{\pi/6}^{\pi/2}\right\} = 2\pi[-0.866 + 1 + 0.433]$$

$$P_{rad} = 3.5626$$

$$D_0 = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi(1)}{3.5626} = 3.5273 = 5.4745 \text{ dB}$$

(b) $$U = \frac{\cos(\theta)}{0.866} = 0.5 \Rightarrow \cos\theta = 0.5(0.866) = 0.433, \theta = \cos^{-1}(0.433) = 64.34°$$

$$\Theta_{1r} = 2(64.34) = 128.68° = 2.246 \text{ rad} = \Theta_{2r}$$

$$D_0 \simeq \frac{4\pi}{\Theta_{1r}\Theta_{2r}} = \frac{4\pi}{(2.246)^2} = 2.4912 = 3.9641 \text{ dB}$$

2.19. (a) 35 dB

(b) $$20\log_{10}\left|\frac{E_{max}}{E_s}\right| = 35,\ \log_{10}\left|\frac{E_{max}}{E_s}\right| = \frac{35}{20} = 1.75$$

$$\left|\frac{E_{max}}{E_s}\right| = 10^{1.75} = 56.234$$

2.20. (a) $U = \sin\theta, U_{max} = 1, \quad P_{rad} = \int_0^{2\pi}\int_0^{\pi} U \sin\theta \, d\theta \, d\phi$

$$= \int_0^{2\pi}\int_0^{\pi} \sin^2\theta \, d\theta \, d\phi = \pi^2$$

$$D_0 = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi}{\pi^2} = \frac{4}{\pi} = 1.2732$$

(b) HPBW $= 120°, 2\pi/3$
The directivity based on (2-33a) is equal to,

$$D_0 = \frac{101}{120° - 0.0027(120°)^2} = 1.2451$$

while that based on (2-33b) is equal to,

$$D_0 = -172.4 + 191\sqrt{0.818 + \frac{1}{120°}} = 1.2245$$

(c) Computer Program: $D_0 = 1.2732$

2.21. (a) $U = \sin^3\theta, U_{max} = 1, P_{rad} = \int_0^{2\pi}\int_0^{\pi} \sin^4\theta \, d\theta \, d\phi = \frac{3}{4}\pi^2$

$$D_0 = \frac{4\pi}{\frac{3}{4}\pi^2} = \frac{16}{3\pi} = 1.6976$$

(b) HPBW $= 74.93°$
From (2-33a), $D_0 = \dfrac{101}{(74.93°) - 0.0027(74.93)°)^2} = 1.68971$
From (2-33b), $D_0 = -172.4 + 191\sqrt{0.818 + \dfrac{1}{74.93°}} = 1.75029$

(c) Computer program: $D_0 = 1.693$
The value of $D_0 = 1.693$ is similar to that of (4-91) or 1.643

2.22. (a) $U = J_1{}^2(\text{ka} \sin\theta)$,
$a = \lambda/10, \text{ka} \sin\theta = \dfrac{\pi}{5} \sin\theta$. HPBW $= 93.10°$
From (2-33a): $D_0 = 101/[(93.10) - 0.0027(93.10)^2] = 1.449120$
From (2-33b): $D_0 = -172.4 + 191\sqrt{0.818 + \dfrac{1}{93.10}} = 1.477271$

$a = \lambda/20, ka \sin\theta = \dfrac{\pi}{10} \sin\theta$, HPBW $= 91.10°$

From (2-33a), $D_0 = 1.47033$; From (2-33b), $D_0 = 1.502$

(b) $a = \dfrac{\lambda}{10}$: $P_{rad} = \displaystyle\int_0^{2\pi}\int_0^{\pi} J_1^2(\text{ka} \sin\theta) \cdot \sin\theta \, d\theta \, d\phi = 0.7638045$

$$U_{max} = 0.0893, \; D_0 = \frac{4\pi(0.0893)}{0.7638045} = 1.469193$$

$$a = \frac{\lambda}{20}:\ P_{rad} = \int_0^{2\pi}\int_0^{\pi} J_1{}^2(\pi/10\sin\theta)\sin\theta\, d\theta\, d\phi = 0.202604$$

$$U_{max} = 0.0240714,\ D_0 = \frac{4\pi(0.0240714)}{0.202604} = 1.49257$$

If the radius of loop is smaller than $\lambda/20$, the directivity approaches 1.5.

2.23. Using the numerical techniques, the directivity for each intensity of Prob. 2.12, with 10° uniform divisions is equal to for $U = \sin\theta\sin\phi$:

(a) Midpoint: $D_0 = \dfrac{4\pi U_{max}}{P_{rad}}$

$$U_{max} = 1:\ P_{rad} = \frac{\pi}{18}\left(\frac{\pi}{18}\right)\sum_{j=1}^{18}\sin\phi_j\sum_{i=1}^{18}\sin^2\theta_i$$

$$\theta_i = \frac{\pi}{36} + (i-1)\frac{\pi}{18},\ i = 1, 2, 3, ..., 18$$

$$\phi_j = \frac{\pi}{36} + (j-1)\frac{\pi}{18},\ j = 1, 2, 3, ..., 18$$

$$P_{rad} = \left(\frac{\pi}{18}\right)^2(11.38656)(8.9924) = 3.119$$

$$D_0 = \frac{4\pi(1)}{3.119} = 4.03 = 6.05\text{ dB}$$

(b) Trailing edge of each division:

$$\text{Trailing edge: } \theta_i = i(\pi/18), i = 1, 2, 3, \dots, 18$$

$$\phi_j = j(\pi/18), j = 1, 2, 3, \dots, 18$$

$$P_{rad} = \left(\frac{\pi}{18}\right)^2(11.25640)(8.96985) = 3.076$$

$$D_0 = \frac{4\pi(1)}{3.119} = 4.09 = 6.11\text{ dB}$$

In a similar manner:

$U = \sin\theta\sin^2\phi$;
(a) $P_{rad} = 2.463 \Rightarrow D_0 = 5.10 = 7.07$ dB
(b) $P_{rad} = 2.451 \Rightarrow D_0 = 5.13 = 7.10$ dB

$U = \sin\theta\sin^3\phi$;
(a) $P_{rad} = 2.092 \Rightarrow D_0 = 6.01 = 7.79$ dB
(b) $P_{rad} = 2.086 \Rightarrow D_0 = 6.02 = 7.80$ dB

$U = \sin^2\theta\sin\phi$;
(a) $P_{rad} = 2.469 \Rightarrow D_0 = 4.74 = 6.76$ dB
(b) $P_{rad} = 2.618 \Rightarrow D_0 = 4.80 = 6.81$ dB

$U = \sin^2\theta\sin^2\phi$;
(a) $P_{rad} = 2.092 \Rightarrow D_0 = 6.01 = 7.79$ dB
(b) $P_{rad} = 2.086 \Rightarrow D_0 = 6.02 = 7.80$ dB

$U = \sin^2\theta\sin^3\phi$;
(a) $P_{rad} = 1.777 \Rightarrow D_0 = 7.07 = 8.49$ dB
(b) $P_{rad} = 1.775 \Rightarrow D_0 = 7.08 = 8.50$ dB

2.24. Using the computer program **Directivity** of Chapter 2, the directivities for each radiation intensity of Problem 2.12 are equal to:

(a) $U = \sin\theta \sin\phi$; $P_{rad} = 3.1318$

$$U_{max} = 1; \quad D_0 = \frac{4\pi \cdot U_{max}}{3.1318} = 4.0125 \Rightarrow 6.034 \text{ dB}$$

(b) $U = \sin\theta \sin^2\phi$; $P_{rad} = 2.4590$

$$U_{max} = 1; \quad D_0 = \frac{4\pi \cdot 1}{2.4590} = 5.110358 \Rightarrow 7.0845 \text{ dB}$$

(c) $U = \sin\theta \sin^3\phi$; $P_{rad} = 2.0870$

$$U_{max} = 1; \quad D_0 = \frac{4\pi \cdot 1}{2.0870} = 6.02124 \Rightarrow 7.80 \text{ dB}$$

(d) $U = \sin^2\theta \sin\phi$; $P_{rad} = 2.6579$

$$U_{max} = 1; \quad D_0 = \frac{4\pi \cdot 1}{2.6579} = 4.72793 \Rightarrow 6.746 \text{ dB}$$

(e) $U = \sin^2\theta \sin^2\phi$; $P_{rad} = 2.0870$

$$U_{max} = 1; \quad D_0 = \frac{4\pi \cdot 1}{2.0870} = 6.02126 \Rightarrow 7.7968 \text{ dB}$$

(f) $U = \sin^2\theta \sin^3\phi$; $P_{rad} = 1.7714$

$$U_{max} = 1; \quad D_0 = \frac{4\pi \cdot 1}{1.7714} = 7.09403 \Rightarrow 8.5089 \text{ dB}$$

2.25. (a) $E|_{max} = \cos\left[\frac{\pi}{4}(\cos\theta - 1)\right]|_{max} = 1$ at $\theta = 0°$.

$$0.707 E_{max} = 0.707 \cdot (1) = \cos\left[\frac{\pi}{4}(\cos\theta_1 - 1)\right]$$

$$\frac{\pi}{4}(\cos\theta_1 - 1) = \pm\frac{\pi}{4} \Rightarrow \theta_1 = \begin{cases} \cos^{-1}(2) = \text{does not exist} \\ \cos^{-1}(0) = 90° = \frac{\pi}{2} \text{ rad.} \end{cases}$$

$$\Theta_{1r} = \Theta_{2r} = 2\left(\frac{\pi}{2}\right) = \pi$$

$$D_0 \simeq \frac{4\pi}{\Theta_{1r}\Theta_{2r}} = \frac{4\pi}{\pi^2} = \frac{4}{\pi} = 1.273 = 1.049 \text{ dB}$$

(b) Using the computer program **Directivity** of Chapter 2

$$D_0 = 2.00789 = 3.027 \text{ dB}$$

Since the pattern is not very narrow, the answer obtained using Kraus' approximate formula is not as accurate.

2.26. (a) $E\Big|_{max} = \cos\left[\frac{\pi}{4}(\cos\theta + 1)\right]\Big|_{max} = 1$ at $\theta = \pi$.

$$0.707 = \cos\left[\frac{\pi}{4}(\cos\theta_1 + 1)\right]$$

$$\frac{\pi}{4}(\cos\theta_1 + 1) = \pm\frac{\pi}{4} \Rightarrow \theta_1 = \begin{cases} \cos^{-1}(-2) \rightarrow \text{does not exist.} \\ \cos^{-1}(0) \rightarrow 90° \rightarrow \frac{\pi}{2} \text{ rad} \end{cases}$$

$$\Theta_{1r} = \Theta_{2r} = 2\left(\frac{\pi}{2}\right) = \pi$$

$$D_0 \simeq \frac{4\pi}{\pi^2} = \frac{4}{\pi} = 1.273 = 1.049 \text{ dB}$$

(b) Computer Program **Directivity**:
$D_0 = 2.00789 = 3.027$ dB

2.27. (a) $P_{rad} = \int_0^{2\pi}\int_0^{\pi/2} U_0 \sin(\pi\sin\theta)\sin\theta\, d\theta\, d\phi = 2\pi U_0 \frac{\pi}{2} J_1(\pi) = U_0\pi^2 J_1(\pi)$

$$D_0 = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi U_0}{U_0\pi^2 J_1(\pi)} = \frac{4}{\pi}\frac{1}{J_1(\pi)} = 4.4735$$

$$\frac{\pi}{2} J_1(\pi) = 0.447$$

(b) Computer program **Directivity**:

$$P_{rad} = \int_0^{2\pi}\int_0^{\pi/2} U_0 \sin(\pi\sin\theta)\sin\theta\, d\theta\, d\phi = 2\pi(0.447)$$

$$D_0 = 4.4735$$

2.28. $E_\phi = C_0 \sin^{1.5}\theta \frac{e^{-jkr}}{r}$

(a) $U_n = |E_\phi|^2 = e_0^2 \sin^3\theta, \Rightarrow U_{n|\max} = C_0^2$

$$P_{rad} = \int_0^{2\pi}\int_0^{\pi} U\sin\theta\, d\theta\, d\phi = 2\pi\int_0^{\pi} C_0^2\sin^3\theta\ \sin\theta\, d\theta = C_0(2\pi)\int_0^{\pi}\sin^4\theta\, d\theta$$

$$\int_0^{\pi}\sin^4\theta\, d\theta = -\frac{\sin^3\theta\cos\theta}{4}\Big|_0^{\pi} + \frac{4-3}{4}\int_0^{\pi}\sin^2\theta\, d\theta = \frac{3}{4}\int_0^{\pi}\sin^2\theta\, d\theta$$

$$= \frac{3}{4}\left[\frac{\theta}{2} - \frac{1}{4}\sin(2\theta)\right]_0^{\pi} = \frac{3}{4}\left(\frac{\pi}{2}\right) = \frac{3\pi}{8}$$

$$P_{rad} = 2\pi C_0^2\left(\frac{3\pi}{8}\right) = \frac{3\pi^2}{4}C_0^2$$

$$D_0 = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi C_0^2}{\frac{3\pi^2}{4}C_0^2} = \frac{16}{3\pi} = 1.69765 = 2.298 \text{ dB}$$

$$\boxed{D_0 = 1.69765 = 2.298 \text{ dB}}$$

(b) $U_n = C_0\sin^3\theta,\ U_n\Big|_{max} = C_0^2$ at $\theta = 90°,\ U_n\Big|_{\theta=\theta_n} = 0.5C_0^2 = \sin^3\theta_h C_0^2$

$$\sin^3\theta_h = 0.5, \theta_h = \sin^{-1}(0.5)^{1/3} = \sin^{-1}(0.7937) = 52.5327°$$

$$\Theta_h = 2(90° - 52.5327°) = 74.935°$$

$$D_0(\text{McDonald}) = \frac{101}{74.935 - 0.0027(74.935)^2} = \frac{101}{59.7738} = \boxed{1.6897 = 2.278 \text{ dB}}$$

$$D_0(\text{McDonald}) = \boxed{1.6897 \text{ dimensionless} = 2.278 \text{ dB}}$$

$$D_0(\text{Pozar}) = -172.4 + 191\sqrt{0.818 + \frac{1}{74.935°}} = -172.4 + 191(0.91178)$$

$$D_0(\text{Pozar}) = -172.4 + 174.1502 = \boxed{1.7502 \text{ dimensionless} = 2.431 \text{ dB}}$$

2.29. (a) Using the computer program **Directivity** of Chapter 2.

$$D_0 = 14.0707 \text{ dimensionless} = 11.48 \text{ dB}$$

(b) $U|_{max} = \left[\dfrac{\sin(\pi\sin\theta)}{\pi\sin\theta}\right]^2_{max} = 1 \quad \text{when } \theta = 0°.$

$$U = \frac{1}{2}U_{max} = \frac{1}{2}(1) = \left[\frac{\sin(\pi\sin\theta_1)}{\pi\sin\theta_1}\right]^2$$

Iteratively we obtain $\theta_1 = 26.3°$. Therefore

$$\Theta_{1d} = \Theta_{2d} = 2(26.3°) = 52.6°.$$

and $D_0 \simeq \dfrac{41,253}{(52.6)^2} = 14.91$ dimensionless $= 11.73$ dB using the Kraus' formula

(c) For Tai and Pereira's formula

$$D_0 = \frac{72,815}{2\cdot\Theta_{1d}^2} = \frac{72,815}{2(52.6)^2} = 13.16 \text{ dimensionless} = 11.19 \text{ dB}$$

2.30. $U = \dfrac{1}{2\eta}|E|^2 = \dfrac{1}{2\eta}\sin\theta\cos^2\phi \Rightarrow U_{max} = \dfrac{1}{2\eta}$

(a) $P_{rad} = 2\cdot\displaystyle\int_0^{\pi/2}\int_0^{\pi}\frac{1}{2\eta}\sin^2\theta\cos^2\phi\, d\theta\, d\phi = \frac{1}{\eta}\left(\frac{\pi}{4}\right)\left(\frac{\pi}{2}\right) = \frac{\pi^2}{8\eta}$

$$D_0 = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi\left(\dfrac{1}{2\eta}\right)}{\dfrac{\pi^2}{8\eta}} = \frac{16}{\pi} = 5.09 = 7.07 \text{ dB}$$

(b) $U_{max} = \dfrac{1}{2\eta}$ at $\theta = \pi/2, \phi = 0$

In the elevation plane through the maximum $\phi = 0$ and $U = \dfrac{1}{2\eta}\sin\theta$, the 3-dB point occurs when

$$U = 0.5U_{max} = 0.5\left(\frac{1}{2\eta}\right) = \frac{1}{2\eta}\sin\theta_1 \Rightarrow \theta_1 = \sin^{-1}(0.5) = 30°$$

Therefore $\Theta_{1d} = 2(90 - 30) = 120°$

In the azimuth plane through the maximum $\theta = \pi/2$ and $U = \dfrac{1}{2\eta}\cos^2\phi$, the 3-dB point occurs when $U = 0.5U_{max} = 0.5\left(\dfrac{1}{2\eta}\right) = \dfrac{1}{2\eta}\cos^2\theta_1 \Rightarrow \phi_1 = \cos^{-1}(0.707) = 45°$

$$\Theta_{2d} = 2(90° - 45°) = 90°$$

Therefore using Kraus' formula: $D_0 \simeq \dfrac{41,253}{120(90)} = 3.82$ dimensionless $= 5.82$ dB

(c) Using Tai and Pereira's formula:

$$D_0 \simeq \frac{72,815}{\Theta_{1d}^2 + \Theta_{2d}^2} = \frac{72,815}{(120)^2 + (90)^2} = 3.24 \text{ dimensionless} = 5.10 \text{ dB}$$

(d) Using the computer program **Directivity** of Chapter 2.

$$D_0 = 5.16425 = 7.13 \text{ dB}$$

2.31. $U = \left[\dfrac{J_1(ka\sin\theta)}{\sin\theta}\right]^2 = (ka)^2\left[\dfrac{J_1(ka\sin\theta)}{ka\sin\theta}\right]^2 = U_0\left[\dfrac{J_1(ka\sin\theta)}{ka\sin\theta}\right]^2$

(a) $U_{\max} = U_0\left(\dfrac{1}{2}\right)^2 = \dfrac{U_0}{4}$ and it occurs when $ka\sin\theta = 0 \Rightarrow \theta = 0°$. The 3-dB point is obtained using

$$U = \frac{1}{2}U_{\max} = \frac{U_0}{8} = U_0\left[\frac{J_1(ka\sin\theta)}{ka\sin\theta}\right]^2 \Rightarrow \frac{J_1(ka\sin\theta)}{ka\sin\theta} = 0.3535$$

with the aid of the $J_1(x)/x$ of Appendix V.

$$x = ka\sin\theta_1 = 1.61 \Rightarrow \theta_1 = \sin^{-1}(1.61/2\pi) = 14.847°$$
$$\Rightarrow \Theta_{1r} = 29.694°$$

(b) Since $\Theta_{1r} = \Theta_{2r} = 29.694°$, the directivity using Kraus' formula is equal to

$$D_0 \simeq \frac{41,253}{(29.694)^2} = 46.79 \text{ dimensionless} = 16.70 \text{ dB}$$

2.32. $G_0 = 16 \text{ dB} \Rightarrow 16 = 10\log_{10} G_0 \text{ (dimensionless)} \Rightarrow G_0\text{(dimensionless)} = 10^{1.6} = 39.81$

$r = 100 \text{ meters} = 10,000 \text{ cm} = 10^4 \text{ cm}$

$P_{\text{rad}} = e_{cd}P_{in} = (1)P_{in} = 8 \text{ watts}$

$f = 1{,}900 \text{ MHz} \Rightarrow \lambda = 30\times10^9/1.9\times10^9 = 15.789 \text{ cm}$

(a)
$$W_0 = \frac{P_{\text{rad}}}{4\pi r^2} = \frac{8}{4\pi(10^4)^2} = \frac{8}{4\pi\times10^8}$$
$$= \frac{2}{\pi}\times10^{-8} = 0.6366\times10^{-8} \text{ Watts/cm}^2$$
$$W_0 = 0.6366\times10^{-8} = 6.366\times10^{-9} \text{ Watts/cm}^2$$
$$W_{\max} = W_0G_0(\text{dim}) = 6.366\times10^{-9}(39.81) = 253.438\times10^{-9}.$$

$$\boxed{W_{\max} = 253.438\times10^{-9} \text{ Watts/cm}^2}$$

(b) $D_0(\lambda/4 \text{ monopole}) = 1.643$

$$A_{em} = \frac{\lambda^2}{4\pi}D_0 = \frac{\lambda^2}{4\pi}(1.643) = \frac{1.643(15.789)^2}{4\pi} = 32.5938 \text{ cm}^2$$
$$A_{em} = 32.5938 \text{ cm}^2$$
$$P(\text{received}) = W_{\max}A_{em} = (253.438\times10^{-9})(32.5938)$$

$$\boxed{P(\text{received}) = 8.2606\times10^{-6} \text{ Watts}}$$

2.33. (a) Linear because $\Delta\phi = 0$.

(b) Linear because $\Delta\phi = 0$.

(c) Circular because
1. $E_x = E_y$
2. $\Delta\phi = \pi/2$.
CCW because E_y leads E_x. AR = 1, $\tau = 90°$

(d) Circular because
1. $E_x = E_y$
2. $\Delta\phi = -\pi/2$
CW because E_y lags E_x. AR = 1, $\tau = 90°$

(e) Elliptical because $\Delta\phi$ is not odd multiples of $\pi/2$. CCW because E_y leads E_x.
AR = OA/OB
Letting $E_x = E_y = E_0$

$$\left.\begin{aligned} \text{OA} &= E_0[0.5(1+1+\sqrt{2})]^{1/2} = 1.30656E_0 \\ \text{OB} &= E_0[0.5(1+1+\sqrt{2})]^{1/2} = 0.541196E_0 \end{aligned}\right\} \Rightarrow \text{AR} = \frac{1.30656}{0.541196} = 2.414$$

$$\tau = 90° - \frac{1}{2}\tan^{-1}\left[\frac{2(1)\cos(45°)}{1-1}\right] = 90° - \frac{1}{2}\tan^{-1}\left(\frac{1.414}{0}\right)$$

$$= 90° - \frac{1}{2}(90°) = 45°$$

(f) Elliptical because $\Delta\phi$ is not odd multiples of $\pi/2$. CW because E_y lags E_x.

$$\text{From above}\ \left.\begin{aligned} \text{OA} &= 1.30656E_0 \\ \text{OB} &= 0.541196E_0 \end{aligned}\right\} \Rightarrow \text{AR} = \frac{1.30656}{0.541196} = 2.414$$

From above $\tau = 90° - \frac{1}{2}(90°) = 45°$

(g) Elliptical because
1. $E_x \neq E_y$
2. $\Delta\phi$ is not zero or multiples of π.
CCW because E_y leads E_x.

$$\left.\begin{aligned} \text{OA} &= E_y\{\tfrac{1}{2}[0.25+1+0.75]\}^{1/2} = E_y \\ \text{OB} &= E_y\{\tfrac{1}{2}[0.25+1-0.75]\}^{1/2} = 0.5E_y \end{aligned}\right\} \Rightarrow \text{AR} = \frac{1}{0.5} = 2$$

$$\tau = 90° - \frac{1}{2}\tan^{-1}\left(\frac{0}{-0.75}\right) = 90° - \frac{1}{2}(180°) = 0°$$

(h) Elliptical because
1. $E_x \neq E_y$
2. $\Delta\phi$ is not zero or multiples of π.
CCW because E_y lags E_x.

$$\text{From above}\ \left.\begin{aligned} \text{OA} &= E_y \\ \text{OB} &= 0.5E_y \end{aligned}\right\} \Rightarrow \text{AR} = \frac{1}{0.5} = 2$$

$$\tau = 90° - \frac{1}{2}(180°) = 0°$$

2.34. $\mathcal{E}_x(z,t) = \text{Re}[E_x e^{j(\omega t + kz + \phi_x)}] = E_x\cos(\omega t + kz + \phi_x)$

$\mathcal{E}_y(z,t) = \text{Re}[E_y e^{j(\omega t + kz + \phi_y)}] = E_y\cos(\omega t + kz + \phi_y)$

where E_x and E_y are real positive constants.

Choosing $z = 0$ and letting $\Delta\phi = \phi_y - \phi_x = \phi_y - 0 = \phi$

$$\mathcal{E}_x(t) = E_x \cos(\omega t) \tag{1}$$
$$\mathcal{E}_y(t) = E_y \cos(\omega t + \phi)$$

and

$$\mathcal{E}(t) = \sqrt{\mathcal{E}_x^2 + \mathcal{E}_y^2} = \sqrt{E_x^2 \cos^2(\omega t) + E_y^2 \cos^2(\omega t + \phi)} \tag{2}$$

The maximum and minimum values of (2) are the major and minor axes of the polarization ellipse. Squaring (2) and using the half-angle identity, (2) can be written as

$$\mathcal{E}^2(t) = \frac{1}{2}\{E_x^2 + E_y^2 + E_x^2 \cos(2\omega t) + E_y^2 \cos^2[2(\omega t + \phi)]\} \tag{3}$$

Since E_x and E_y are constants, the maximum and minimum values of (3) occur when $f(t) = E_x{}^2 \cos(2\omega t) + E_y{}^2 \cos[2(\omega t + \phi)]$ is maximum or minimum. These are found by differentiating (4) and setting it equal to zero. Thus

$$\frac{df}{d(2\omega t)} = -E_x{}^2 \sin(2\omega t) - E_y^2 \sin[2(\omega t + \phi)] = 0 \tag{4}$$

or

$$E_x^2 \sin(2\omega t) = -E_y^2 \sin[2(\omega t + \phi)]$$
$$= -E_y^2\{\sin 2\omega t \cos 2\phi + \cos 2\omega t \sin 2\phi\} \tag{5}$$

Dividing (5) by $\cos(2\omega t)$ yields

$$E_x^2 \tan(2\omega t) = -E_y^2[\tan(2\omega t)\cos(2\phi) + \sin(2\phi)]$$

or

$$\tan(2\omega t) = \frac{-E_y^2 \sin(2\phi)}{E_x^2 + E_y^2 \cos(2\phi)}$$

from which we obtain that

$$\cos(2\omega t) = \frac{E_x^2 + E_y^2 \cos(2\phi)}{\pm\rho} \tag{6}$$

$$\cos(2\omega t + 2\phi) = \frac{E_y^2 + E_x^2 \cos(2\phi)}{\pm\rho} \tag{7}$$

where

$$\rho = \sqrt{E_x^4 + E_y^4 + 2E_x^2 E_y^2 \cos(2\phi)} \tag{8}$$

Substituting (6)–(8) into (3) yields

$$\mathcal{E}^2 = \frac{1}{2}\left[E_x^2 + E_y^2 \pm \frac{1}{\rho}(\rho^2)\right]$$

whose maximum value is

$$\mathcal{E}_{\max} = \text{OA} = \left\{\frac{1}{2}[E_x^2 + E_y^2 + (E_x^4 + E_y^4 + 2E_x^2E_y^2\cos 2\phi)^{1/2}]\right\}^{1/2}$$

$$\mathcal{E}_{\max} = \text{OB} = \left\{\frac{1}{2}[E_x^2 + E_y^2 - (E_x^4 + E_y^4 + 2E_x^2E_y^2\cos 2\phi)^{1/2}]\right\}^{1/2}$$

The tilt angle τ can be obtained by expanding (1) and writing the two as

$$\frac{\mathcal{E}_x^2}{E_x^2} - \frac{2\mathcal{E}_x\mathcal{E}_y\cos\phi}{E_xE_y} + \frac{\mathcal{E}_y^2}{E_y^2} = \sin^2\phi \qquad (9)$$

which is the equation of a tilted ellipse. Choosing a coordinate system whose principal axes coincide with the major and minor axes of the tilted ellipse, we can write that

$$\begin{aligned}\mathcal{E}_x &= \mathcal{E}'_x\sin(z) - \mathcal{E}'_y\cos(z)\\ \mathcal{E}_y &= \mathcal{E}'_x\cos(z) + \mathcal{E}'_y\sin(z)\end{aligned} \qquad (10)$$

where $\mathcal{E}'_x$ and $\mathcal{E}'_y$ are the new field values along the new principal axes x', y', z'. Substituting (10) into (9) yields

$$\frac{2\mathcal{E}'_x\mathcal{E}'_y\cos(z)\sin(z)}{E_x^2} - \frac{2\mathcal{E}'_x\mathcal{E}'_y\cos(z)\sin(z)}{E_y^2} - \frac{2\mathcal{E}'_x\mathcal{E}'_y\cos\phi}{E_xE_y}(\sin^2 z - \cos^2 z) = 0$$

which when solved for the tilt angle τ reduces to

$$\tan\left[2\left(\frac{\pi}{2} - \tau\right)\right] = \frac{2E_xE_y\cos\phi}{E_x^2 - E_y^2}$$

or

$$\tau = \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\left(\frac{2E_xE_y\cos\phi}{E_x^2 - E_y^2}\right)$$

For more details on the tilt angle derivation, see J.D. Kraus, *Antennas*, McGraw-Hill, 1950, pp. 464–476.

2.35. (a) $\hat{\rho}_w = \hat{a}_x\cos\phi_1 + \hat{a}_y\sin\phi_1$

$\hat{\rho}_a = \hat{a}_x\cos\phi_2 + \hat{a}_y\sin\phi_2$

$$\begin{aligned}\text{PLF} &= |\hat{\rho}_w\cdot\hat{\rho}_a|^2 = |(\hat{a}_x\cos\phi_1 + \hat{a}_y\sin\phi_1)\cdot(\hat{a}_x\cos\phi_2 + \hat{a}_y\sin\phi_2)|^2\\ &= |\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2|^2 = |\cos(\phi_1 - \phi_2)|^2\end{aligned}$$

(b) $\hat{\rho}_w = \hat{a}_x \sin\theta_1 \cos\phi_1 + \hat{a}_y \sin\theta_1 \sin\phi_1 + \hat{a}_z \cos\theta_1$

$\hat{\rho}_a = \hat{a}_x \sin\theta_2 \cos\phi_2 + \hat{a}_y \sin\theta_2 \sin\phi_2 + \hat{a}_z \cos\theta_2$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = |\sin\theta_1 \cos\phi_1 \sin\theta_2 \cos\phi_2 + \sin\theta_1 \sin\phi_1 \sin\theta_2 \cdot \sin\phi_2 + \cos\theta_1 \cdot \cos\theta_2|^2$$

$$\text{PLF} = |\sin\theta_1 \cdot \sin\theta_2(\cos\phi_1 \cdot \cos\phi_2 + \sin\phi_1 \sin\phi_2) + \cos\theta_1 \cos\theta_2|^2$$

$$\text{PLF} = |\sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2) + \cos\theta_1 \cos\theta_2|^2$$

2.36. Assuming electric field is x-polarized

(a) $\underline{E}_w = \hat{a}_x E_1 e^{-jkz} \Rightarrow \hat{\rho}_w = \hat{a}_x$

$$\underline{E}_a = (\hat{a}_\theta - j\hat{a}_\phi)E_0 f(r,\theta,\phi) \Rightarrow \hat{\rho}_a = \left(\frac{\hat{a}_\theta - j\hat{a}_\phi}{\sqrt{2}}\right)$$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \frac{1}{2}|\hat{a}_x \cdot \hat{a}_\theta - j\hat{a}_x \cdot \hat{a}_\phi|^2$$

since $\hat{a}_\theta = \hat{a}_x \cos\theta \cos\phi + \hat{a}_y \cos\theta \sin\phi - \hat{a}_z \sin\theta$

$\hat{a}_\phi = -\hat{a}_x \sin\phi + \hat{a}_y \cos\phi$

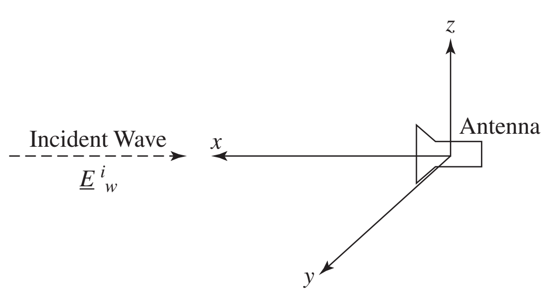
$$\text{PLF} = \tfrac{1}{2}(\cos^2\theta \cos^2\phi + \sin^2\phi)$$

(b) when $\underline{E}_a = (\hat{a}_\theta + j\hat{a}_\theta)E_0 f(r,\theta,\phi)$, PLF is also

$$\text{PLF} = \tfrac{1}{2}(\cos^2\theta \cos^2\phi + \sin^2\phi)$$

A more general, but also more complex, expression can be derived when the incident electric field is of the form $\underline{E}_w = (a\hat{a}_x + b\hat{a}_y)e^{-jkz}$ where a, b are real constants. It can be shown (using the same procedure) that

$$\text{PLF} = \frac{1}{\sqrt{2(a^2+b^2)}}[(a\cos\theta\cos\phi + b\sin\theta\sin\phi)^2 + (a\sin\phi - b\cos\phi)^2]^{1/2}$$



2.37. (a) $\underline{E}_w = E_0(j\hat{a}_y + 3\hat{a}_z)e^{+jkx}$

1. **Elliptical polarization; AR = $\frac{3}{1}$ = 3; Left Hand (CCW)**
   a. 2 components orthogonal to direction of propagation
   b. Not of same magnitude
   c. 90° phase difference between them
   d. y component is leading the z component *or* z component is lagging the y component

(b) $\underline{E}_a = E_a(\hat{a}_y + 2\hat{a}_z)e^{-jkx}$

1. **Liner polarization; AR = ∞; No rotation**
   a. 2 components orthogonal to direction of propagation.
   b. Not of same magnitude
   c. 0° phase difference between them,

(c) $\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2$

$$\underline{E}_w = E_0(j\hat{a}_y + 3\hat{a}_z)e^{+jkx} = E_0 \underbrace{\left(\frac{j\hat{a}_y + 3\hat{a}_z}{\sqrt{10}}\right)}_{\hat{\rho}_w} \sqrt{10}e^{+jkx}$$

$$\hat{\rho}_w = \left(\frac{j\hat{a}_y + 3\hat{a}_z}{\sqrt{10}}\right)$$

$$\underline{E}_a = E_a(\hat{a}_y + 2\hat{a}_z)e^{-jkx} = E_0 \underbrace{\left(\frac{\hat{a}_y + 2\hat{a}_z}{\sqrt{5}}\right)}_{\hat{\rho}_a} \sqrt{5}e^{-jkx}$$

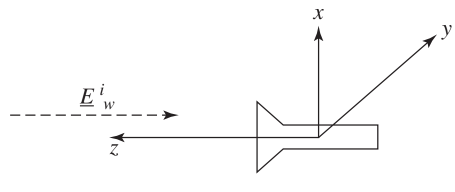
$$\hat{\rho}_a = \left(\frac{\hat{a}_y + 2\hat{a}_z}{\sqrt{5}}\right)$$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\frac{(j\hat{a}_y + 3\hat{a}_z)}{\sqrt{10}} \cdot \frac{(\hat{a}_y + 2\hat{a}_z)}{\sqrt{5}}\right|^2 = \frac{|j+6|^2}{50} = \frac{37}{50}$$

$$\text{PLF} = \frac{37}{50} = \boxed{0.740 = -1.31\text{ dB}}$$

2.38. $\underline{E}_w^i = (\hat{a}_x + j\hat{a}_y)E_0 e^{+jkz}$

$$\underline{E}_a = (\hat{a}_x + 2\hat{a}_y)E_1 \frac{e^{-jkr}}{r}\bigg|_{\substack{\theta = 0° \\ z \text{ axis}}} = (\hat{a}_x + 2\hat{a}_y)E_1 \frac{e^{-jkz}}{z}$$



(a) $\underline{E}_w^i = \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right)\sqrt{2}E_0 e^{+jkz}$

Circular: 2 components, same amplitude, 90° phase difference

(b) **Clockwise** (y component is leading the x component)

(c) $\underline{E}_a = \left(\frac{\hat{a}_x + 2\hat{a}_y}{\sqrt{5}}\right)\sqrt{5}E_1 \frac{e^{-jkz}}{z}$

Linear: 2 components, 0° phase difference

(d) No rotation

(e) $\hat{\rho}_w = \left(\dfrac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right), \quad \hat{\rho}_a = \left(\dfrac{\hat{a}_x + 2\hat{a}_y}{\sqrt{5}}\right)$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left[\left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right) \cdot \left(\frac{\hat{a}_x + 2\hat{a}_y}{\sqrt{5}}\right)\right]^2 = \frac{|1 + j2|^2}{10} = \frac{5}{10}$$

$$\text{PLF} = \frac{5}{10} = 0.5 = 10\log_{10}(0.5) = -3 \text{ dB}$$

2.39. (a) $\underline{E}_w = (4\hat{a}_z + j2\hat{a}_x)E_w \dfrac{e^{+jky}}{y} = \underbrace{\left(\dfrac{4\hat{a}_z + j2\hat{a}_x}{\sqrt{20}}\right)}_{\hat{\rho}_w} \sqrt{20}E_w \dfrac{e^{+jky}}{y}$

- **Elliptical** (2 components, not of same magnitude, 90° phase difference)

(b) **CW**; x-components leads z-component by 90°; rotate x into z while looking (observing) in the -y direction (from behind the wave).

(c) **AR** $= \dfrac{4}{2} = 2$

(d) $\hat{\rho}_w = \left(\dfrac{4\hat{a}_z + j2\hat{a}_x}{\sqrt{20}}\right); \ \hat{\rho}_a = \hat{a}_z$

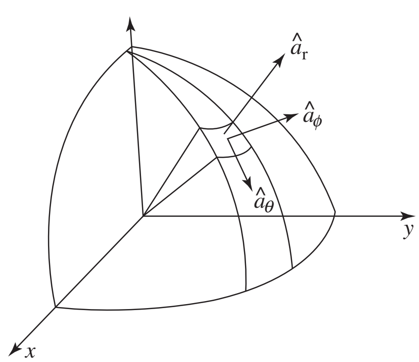
$$\textbf{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\left(\frac{4\hat{a}_z + j2\hat{a}_x}{\sqrt{20}}\right) \cdot \hat{a}_z\right|^2 = \frac{16}{20} = 0.8 = 10\log_{10}(0.8)$$

$$\text{PLF} = 0.8 = -0.969 \text{ dB}$$

2.40. (a) $\underline{E}_a = E_0(j\hat{a}_\theta + 2\hat{a}_\phi)f_0(\theta_0, \phi_0)\dfrac{e^{-jkr}}{r} = E_0 \underbrace{\left(\dfrac{j\hat{a}_\theta + 2\hat{a}_\phi}{\sqrt{5}}\right)}_{\hat{\rho}_a} \sqrt{5}f_0(\theta_0, \phi_0)\dfrac{e^{-jkr}}{r}$

$$\hat{\rho}_a = \left(\frac{j\hat{a}_\theta + 2\hat{a}_\phi}{\sqrt{5}}\right)$$

Elliptical, CW



(b) $\underline{E}_w = E_1(2\hat{a}_\theta + j\hat{a}_\phi)f_1(\theta_0, \phi_0)\dfrac{e^{+jkr}}{r}$

$$= E_1 \underbrace{\left(\frac{2\hat{a}_\theta + j\hat{a}_\phi}{\sqrt{5}}\right)}_{\hat{\rho}_w} \sqrt{5} f_1(\theta_0, \phi_0)\frac{e^{+jkr}}{r}$$

$$\hat{\rho}_w = \left(\frac{2\hat{a}_\theta + j\hat{a}_\phi}{\sqrt{5}}\right)$$

Elliptical, CW

(c) $\text{PLF} = |\hat{\rho}_a \cdot \hat{\rho}_w|^2 = \left|\left(\dfrac{j\hat{a}_\theta + 2\hat{a}_\phi}{\sqrt{5}}\right) \cdot \left(\dfrac{2\hat{a}_\theta + j\hat{a}_\phi}{\sqrt{5}}\right)\right|^2 = \left|\dfrac{2j + j2}{\sqrt{25}}\right|^2 = \left|\dfrac{4j}{\sqrt{25}}\right|^2$

$$\boxed{\text{PLF} = \frac{16}{25} = 0.64 = -1.938 \text{ dB}}$$

2.41. (a) $\underline{E}_w = E_0(\hat{a}_x \pm j\hat{a}_y)e^{-jkz} \Rightarrow \hat{\rho}_w = \dfrac{1}{\sqrt{2}}(\hat{a}_x \pm j\hat{a}_y)$

$$\underline{E}_a \simeq E_1(\hat{a}_\theta - j\hat{a}_\phi)f(r, \theta, \phi) \Rightarrow \hat{\rho}_a = \frac{1}{\sqrt{2}}(\hat{a}_\theta - j\hat{a}_\phi)$$

$$\text{PLF} = \frac{1}{2}\left|(\hat{a}_x \pm j\hat{a}_y) \cdot (\hat{a}_\theta - j\hat{a}_\phi)\right|^2 = \frac{1}{2}\left|(\hat{a}_x \cdot \hat{a}_\theta \pm \hat{a}_y \cdot \hat{a}_\phi) - j(\hat{a}_x\hat{a}_\phi \mp \hat{a}_y\hat{a}_\theta)\right|^2$$

Converting the spherical unit vectors to rectangular, as it was done in Problem 2.35, leads to

$$\text{PLF} = \frac{1}{2}(\cos\theta \pm 1)^2$$

(b) When

$$\underline{E}_w = E_0(\hat{a}_x \pm j\hat{a}_y)e^{-jkz}$$
$$\underline{E}_a \simeq E_1(\hat{a}_\theta + j\hat{a}_\phi)f(r, \theta, \phi)$$

the PLF is equal to

$$\text{PLF} = \frac{1}{2}(\cos\theta \mp 1)^2$$

2.42. $\underline{E}_w = (\hat{a}_\theta \cos\phi - \hat{a}_\phi \sin\phi \cos\theta)f(r, \theta, \phi)$ or

$$\underline{E}_w = \left[\frac{\hat{a}_\theta \cos\phi - \hat{a}_\phi \sin\phi\cos\theta}{\sqrt{\cos^2\phi + \sin^2\phi\cos^2\theta}}\right]\sqrt{\cos^2\phi + \sin^2\phi\cos^2\theta} \cdot f(r, \theta, \phi)$$

Thus $\hat{\rho}_w = \dfrac{\hat{a}_\theta \cos\phi - \hat{a}_\phi \sin\phi\cos\theta}{\sqrt{\cos^2\phi + \sin^2\phi\cos^2\theta}}$

and

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left| \left(\frac{\hat{a}_\theta \cos\phi - \hat{a}_\phi \sin\phi \cos\theta}{\sqrt{\cos^2\phi + \sin^2\phi \cos^2\theta}} \right) \cdot \hat{a}_x \right|^2$$

Transforming the rectangular unit vector to spherical using

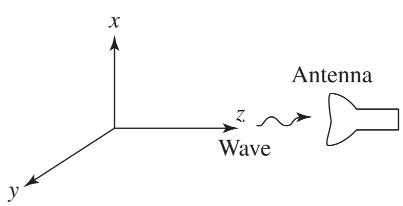
$$\hat{a}_x = \hat{a}_r \sin\theta \cos\phi + \hat{a}_\theta \cos\theta \cos\phi - \hat{a}_\phi \sin\phi$$

the PLF reduces to

$$\text{PLF} = \frac{\cos^2\theta}{\cos^2\phi + \sin^2\phi \cos^2\theta}$$

The same answer is obtained by transforming the spherical unit vectors to rectangular, as was done in Prob. 2.35.

2.43. $\underline{E}_a \simeq (2\hat{a}_x \pm j\hat{a}_y) f(r,\theta,\phi) = \left(\frac{2\hat{a}_x \pm j\hat{a}_y}{\sqrt{5}} \right) \sqrt{5} f(r,\theta,\phi)$



(a) $\hat{\rho}_w = \left(\frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}} \right) \Rightarrow$ Wave is Right Hand (RH)

$$\hat{\rho}_a = \left(\frac{2\hat{a}_x \pm j\hat{a}_y}{\sqrt{5}} \right)$$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2$$

$$= \begin{cases} \frac{9}{10} = -0.4576 \text{ dB using the } + \text{ sign} & \text{(Antenna is LH in receiving mode and RH in transmitting)} \\ \frac{1}{10} = -10 \text{ dB using the } - \text{ sign} & \text{(Antenna is RH in receiving mode and LH in transmitting)} \end{cases}$$

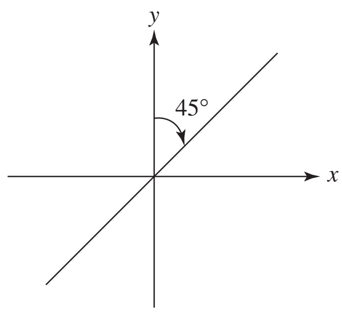
(b) $\hat{\rho}_w = \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}} \right) \Rightarrow$ Wave is Left Hand (LH)

$$\hat{\rho}_a = \left(\frac{2\hat{a}_x \pm j\hat{a}_y}{\sqrt{5}} \right)$$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2$$

$$= \begin{cases} \frac{1}{10} = -10 \text{ dB using the } + \text{ sign} & \text{(Antenna is LH in receiving mode and RH in transmitting)} \\ \frac{9}{10} = -0.4576 \text{ dB using the } - \text{ sign} & \text{(Antenna is RH in receiving mode and LH in transmitting)} \end{cases}$$

2.44. For $\hat{\rho}_w$



$$\hat{\rho}_w = \frac{\hat{a}_x + \hat{a}_y}{\sqrt{2}}; \quad \text{PLF} = \left| \frac{\hat{a}_x + \hat{a}_y}{\sqrt{2}} \cdot \frac{4\hat{a}_x + j\hat{a}_y}{\sqrt{17}} \right|^2$$

$$\text{PLF} = \frac{1}{34} |(\hat{a}_x \cdot 4\hat{a}_x) + (\hat{a}_y \cdot j\hat{a}_y)|^2 = \frac{1}{34} |4 + j|^2 = 0.5 \text{ dimensionless} = -3 \text{ dB}$$

2.45. (a) RHCP; $\hat{\rho}_a = \dfrac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}$

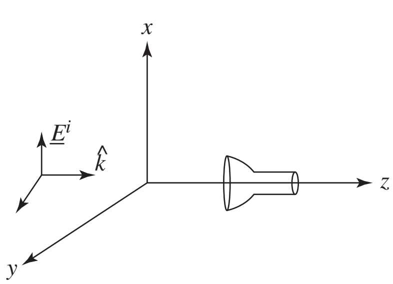
$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left| \frac{2\hat{a}_x + j\hat{a}_y}{\sqrt{5}} \cdot \frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}} \right|^2 = 0.9 \text{ dimensionless} = -0.46 \text{ dB}$$

(b) LHCP; $\hat{\rho}_a = \dfrac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left| \frac{2\hat{a}_x + j\hat{a}_y}{\sqrt{5}} \cdot \frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}} \right|^2 = 0.1 \text{ dimensionless} = -10.0 \text{ dB}$$

2.46. $\underline{E}^i = (\hat{a}_x - j\hat{a}_y)E_0 e^{-jkz} = \left(\dfrac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}} \right) \sqrt{2} E_0 e^{-jkz}$

$$\hat{\rho}_w = \frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}} \qquad \text{CW}$$



(a) $\underline{E}^a = (\hat{a}_x + j\hat{a}_y)E_1 e^{+jkz}$

$$= \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right)\sqrt{2}E_1 e^{+jkz}$$

$$\hat{\rho}_a = \frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}} \quad \text{CW}$$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\left(\frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}\right)\cdot\left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right)\right|^2 = \left(\frac{1-j^2}{2}\right)^2 = 1$$

$\text{PLF} = 1 = 0$ dB

(b) $\underline{E}^a = \left(\frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}\right)\sqrt{2}E_1 e^{+jkz}$

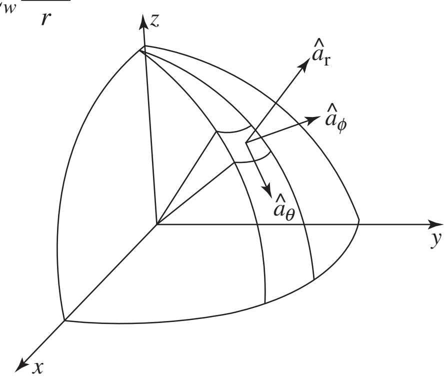
$$\hat{\rho}_a = \frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}$$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\left(\frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}\right)\cdot\left(\frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}\right)\right|^2 = \left|\frac{1+j^2}{2}\right|^2 = 0$$

$\text{PLF} = 0 = -\infty$ dB

2.47. $\underline{E}_a = (2\hat{a}_\theta + j4\hat{a}_\phi)E_a \frac{e^{-jkr}}{r} = \underbrace{\left(\frac{2\hat{a}_\theta + j4\hat{a}_\phi}{\sqrt{20}}\right)}_{\hat{\rho}_a} 20E_a \frac{e^{-jkr}}{r}$

$$\underline{E}_w = (j4\hat{a}_\theta + 2\hat{a}_\phi)E_w \frac{e^{+jkr}}{r} = \underbrace{\left(\frac{j4\hat{a}_\theta + 2\hat{a}_\phi}{\sqrt{20}}\right)}_{\hat{\rho}_w} 20E_w \frac{e^{+jkr}}{r}$$



Antenna

a. Elliptical
b. CCW
c. **AR** $= \frac{4}{2} = 2$

Wave

d. Elliptical
e. CCW
f. **AR** $= \frac{4}{2} = 2$
g. $\text{PLF} = |\hat{\rho}_a \cdot \hat{\rho}_w|^2 = \left|\left(\frac{2\hat{a}_\theta + j4\hat{a}_\phi}{\sqrt{20}}\right)\cdot\left(\frac{j4\hat{a}_\theta + 2\hat{a}_\phi}{\sqrt{20}}\right)\right|^2$

$$= \left|\frac{j8 + j8}{20}\right| = \left|\frac{16}{20}\right|^2 = (0.8)^2 = 0.64$$

PLF = 0.64 dimensionless = −1.9382 dB

2.48. $\underline{E}^i = \hat{a}_x E_0 e^{-jkz}, \hat{\rho}_w = \hat{a}_x$

$$\underline{E}^a = (\hat{a}_x + j\hat{a}_y)E_1 e^{+jkz} = \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right)\sqrt{2}E_1 e^{+jkz}$$

$$\hat{\rho}_a = \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right)$$

(a) $A_{em} = \dfrac{\lambda^2}{4\pi} e_o D_0 |\hat{\rho}_a \cdot \hat{\rho}_w|^2 = \dfrac{\lambda^2}{4\pi} G_0 |\hat{\rho}_a \cdot \hat{\rho}_w|^2$

$(e_o D_0 = G_0)$

$$\text{At 10 GHz} \Rightarrow \lambda = \frac{c}{f} = \frac{3\times 10^8}{10\times 10^9} = \frac{3\times 10^8}{10^{10}} = 3\times 10^{-2}$$

$$G_0 = 10 = 10\log_{10} G_0(\text{dim}) \Rightarrow G_0(\text{dim}) = 10^1 = 10$$

$$A_{em} = \frac{\lambda^2}{4\pi} G_0 |\hat{\rho}_a \cdot \hat{\rho}_w|^2 = \frac{(3\times 10^{-2})^2}{4\pi}(10)\left|\hat{a}_x \cdot \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right)\right|^2$$

$$= \frac{9\times 10^{-4}}{4\pi}(10)\left(\frac{1}{2}\right) = \frac{9\times 10^{-3}}{4\pi}\left(\frac{1}{2}\right) = (0.7162\times 10^{-3})\left(\frac{1}{2}\right)$$

$$A_{em} = 0.3581\times 10^{-3}\ \text{m}^2$$

(b) $P_T = A_{em} W^i = (0.3581\times 10^{-3})(10\times 10^{-3}) = 3.581\times 10^{-6}$ Watts

$P_T = 3.581\times 10^{-6}$ Watts

2.49. $\underline{E}_w = \hat{a}_z E_w \dfrac{e^{+jky}}{y}, \hat{\rho}_w = \hat{a}_z, E_a = -\hat{a}_z E_a \dfrac{e^{-jky}}{y}, \hat{\rho}_a = -\hat{a}_z, W_{\text{inc}} = 100\times 10^{-3}\dfrac{W}{\text{cm}^2}$

(a) $\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = |-\hat{a}_z \cdot \hat{a}_z|^2 = 1 = 0$ dB

(b) For the $\lambda/2$ dipole ($Z_a = 73 + j42.5$) with a loss resistance R_L of 5 ohms:

$$U_n = (E_{\theta n})^2 = (\sin^{1.3}\theta)^2 = \sin^3\theta \Rightarrow (U_n)_{\max} = 1$$

$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}}$$

$$P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi} U\sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\left(\int_0^{\pi}\sin^3\theta\sin\theta\, d\theta\right)d\phi = 2\pi\int_0^{\pi}\sin^4\theta\, d\theta$$

$$\int_0^{\pi}\sin^4\theta\, d\theta = -\frac{\sin^3\theta\cos\theta}{4}\bigg|_0^{\pi} + \frac{4-1}{4}\int_0^{\pi}\sin^2\theta\, d\theta = \frac{3}{4}\int_0^{\pi}\sin^2\theta\, d\theta$$

$$\int_0^{\pi}\sin^4\theta\, d\theta = \frac{3}{4}\int_0^{\pi}\sin^2\theta\, d\theta = \frac{3}{4}\left[\frac{\theta}{2} - \frac{1}{4}\sin(2\theta)\right] = \frac{3\pi}{8}$$

$$P_{\text{rad}} = 2\pi \int_0^{\pi} \sin^4\theta\, d\theta = 2\pi\left(\frac{3\pi}{8}\right) = \frac{3\pi^2}{4}$$

$$\therefore D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}} = D_0 = \frac{4\pi(1)}{3\pi^2/4} = \frac{16}{3\pi} = 1.69765 \text{ (dimensionless)} = 2.298 \text{ dB}$$

Using the equivalent circuit of Figure 1.2 with $R_r = 73$ and $R_L = 5$

$$e_{cd} = \frac{R_r}{R_r + R_L} = \frac{73}{73+5} = 0.9359$$

$$\therefore G_0 = e_{cd}D_0 = 0.9359(1.69765) = 1.5888 = 2.011 \text{ dB}$$

$$|\Gamma| = \left|\frac{Z_{in} - Z_c}{Z_{in} + Z_c}\right| = \left|\frac{(Z_a + R_L) - Z_c}{(Z_a + R_L) + Z_c}\right| = \left|\frac{(73 + j42.5 + 5) - 50}{(73 + j42.5 + 5) + 50}\right| = \frac{50.8945}{134.8712} = 0.3774$$

$$|\Gamma|^2 = (0.3774)^2 = 0.1424 \Rightarrow \left(1 - |\Gamma|^2\right) = (1 - 0.1424) = 0.8576$$

$$G_{re0} = e_r G_0 = \left(1 - |\Gamma|^2\right) G_0 = (0.8576)\, 1.5888 = 1.3626 \text{ (dim)} = 1.344 \text{ dB}$$

$$P_{\text{received}} = A_{em}(e_{cd})(1 - |\Gamma|^2)\text{PLF} = \left(\frac{\lambda^2}{4\pi}D_0\right) e_{cd}(1 - |\Gamma|^2)\text{PLF}$$

$$= \frac{\pi^2}{4\pi}\Big[\underbrace{\underbrace{D_0 e_{cd}}_{G_0}(1 - |\Gamma|^2)}_{G_{re0}}\Big](\text{PLF})W_{inc}$$

$$P_{\text{received}} = (W_{inc})\frac{\lambda^2}{4\pi}G_{re}(\text{PLF}), \qquad \lambda = \frac{30\times 10^9}{10\times 10^9} = 3 \text{ cm}$$

$$= \underbrace{(100\times 10^{-3})}_{W_{inc}}\underbrace{\left(\frac{(3)^2}{4\pi}\right)}_{3^2/4\pi}\underbrace{(1.3562)}_{G_{re0}}\underbrace{(1)}_{\text{PLF}} = \frac{10^{-1}(9)(1.3562)}{4\pi}$$

$$P_{\text{received}} = 0.0981 \text{ Watts} = 98.1 \text{ mWatts} = 98.1\times 10^{-3} \text{ Watts}$$

$$P_{\text{received}} = \underbrace{(100\times 10^{-3})}_{W_{\text{inc}}}\underbrace{\left[\frac{(3)^2}{4\pi}\right]}_{3^2/4\pi}\underbrace{(1.3626)}_{G_{re0}}\underbrace{(1)}_{\text{PLF}} = 97.59 \text{ mW} = 97.59\times 10^{-3} \text{ W}$$

2.50. $\underline{E}_a = (2\hat{a}_x \pm j\hat{a}_y)Ee^{-jkz}$

$$\hat{\rho}_a = \frac{2\hat{a}_x \pm j\hat{a}_y}{\sqrt{5}}$$

(a) $\underline{E}_w = \hat{a}_x E_w \Rightarrow \hat{\rho}_w = \hat{a}_x$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\frac{2}{\sqrt{5}}\right|^2 = \frac{4}{5} = 0.8 \text{ dimensionless} = -0.9691 \text{ dB}$$

(b) $\underline{E}_w = \hat{a}_y E_w \Rightarrow \hat{\rho}_w = \hat{a}_y$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\frac{1}{\sqrt{5}}\right|^2 = \frac{1}{5} = 0.2 \text{ dimensionless} = -6.9897 \text{ dB}$$

2.51. (a) $E_y = E_y' + E_y'' = 3\cos\omega t + 2\cos\omega t = 5\cos\omega t$

$$E_x = E_x' + E_x'' = 7\cos\left(\omega t + \frac{\pi}{2}\right) + 3\cos\left(\omega t - \frac{\pi}{2}\right)$$
$$= -7\sin\omega t + 3\sin\omega t = -4\sin\omega t$$
$$\text{AR} = \frac{5}{4} = 1.25$$

(b) At $\omega t = 0, \underline{E} = 5\hat{a}_y$
At $\omega t = \pi/2 \Rightarrow \underline{E} = -4\hat{a}_x \Rightarrow$ Rotation in CCW

2.52. (a) $\text{PLF} = \frac{1}{2}$ independent of $\psi \rightarrow$ must have CP

$\therefore$ AR = 1.

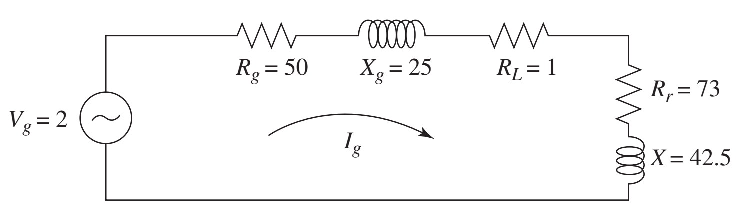
(b) Polarization will be elliptical with major axis aligned with x-axis.
Guess: AR = 2
Verify: $\hat{\rho}_w = (2\hat{a}_x + ja_y)/\sqrt{5}$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\frac{2\cos\psi + j\sin\psi}{\sqrt{5}}\right|^2 = \frac{4\cos^2\psi + \sin^2\psi}{\sqrt{5}}$$

$\psi = 0$: PLF = 0.8
$\psi = 90°$: PLF = 0.2

(c) PLF = 1 at $\psi = 45°$ and $225°$
PLF = 0 at $\psi = 135°$ and $315°$
Polarization must be linear at an angle of 45°
$\therefore$ AR = ∞

2.53.
$$I_g = \frac{2}{(50 + 1 + 73) + j(25 + 42.5)} = \frac{2}{124 + j67.5}$$
$$= (12.442 - j6.7724) \times 10^{-3} = 14.166 \times 10^{-3} \angle -28.56°$$



(a) $P_s = \frac{1}{2}\text{Re}(V_g \cdot I_g^*) = \text{Re}(12.442 + j6.7724) \times 10^{-3} = 12.442 \times 10^{-3}$ W

(b) $P_r = \frac{1}{2}|I_g|^2 R_r = 7.325 \times 10^{-3}$ W

(c) $P_L = \frac{1}{2}|I_g|^2 R_L = 0.1003 \times 10^{-3}$ W

The remaining supplied power is dissipated as heat in the internal resistor of the generator, or

$$P_g = \frac{1}{2}|I_g|^2 R_g = 5.0169 \times 10^{-3} \text{ W}$$