1. COMBINATORIAL ARGUMENTS

1.1. CLASSICAL MODELS

1.1.1. *When rolling n dice, the probability that the sum is even is* $1/2$. No matter what is rolled on the first $n-1$ dice, the last die has three even values and three odd values, so in each case the probability of ending with an even total is $1/2$.

1.1.2. *There are* $\binom{m}{2}\binom{n}{2}$ *rectangles with positive area formed by segments in a grid of m horizontal lines and n vertical lines.* Positive area requires two distinct horizontal boundaries and two distinct vertical boundaries.

1.1.3. *There are* $\binom{r+s}{r}21^r5^s$ *words consisting of r consonants and s vowels.* There are $\binom{r+s}{r}$ ways to allocate the positions to consonants and vowels and then 21^r5^s ways to fill those positions.

1.1.4. *There are* $\binom{33}{3}$ *outcomes of an election with* 30 *voters and four candidates,* $\binom{33}{3}-4\binom{17}{3}$ *with no candidate having more than half of the votes.* If the votes are considered distinct, then there are 4^{30} outcomes. However, votes go into a ballot box, so an outcome is determined just by the number of votes for each candidate. Thus we want the number of nonnegative integer solutions to $x_1+x_2+x_3+x_4=30$, which is $\binom{30+4-1}{4-1}$.

When one candidate receives at least 16 votes, the outcomes are the ways to distribute the remaining 14 votes arbitrarily, since votes are indistinguishable. Only one candidate can have a majority, but that can be any one of the four, so there are $4\binom{17}{3}$ outcomes we exclude.

1.1.5. *For* $n\in\mathbb{N}$, *the expression* $(n^5-5n^3+4n)/120$ *is a integer.* Since $n^5-5n^3+4n=n(n^2-1)(n^2-4)=(n+2)(n+1)n(n-1)(n-2)$, the expression equals $\binom{n+2}{5}$, which is the number of was to choose five objects from a set of size $n+2$. This by definition is an integer.

1.1.6. 13!40! *orderings of a deck of cards such that the spade suit appears consecutively.* There are 13! ways to order the spade suit and 39! ways to order the remaining cards. There are then 40 ways to insert the ordered spade suit among the other cards. Alternatively, condense the spade suit to a single item, order the items in 40! ways, and the expand the spade into 13! orderings of the spade suit.

1.1.7. *The probability of having at least three cards with the same rank in a set of five ordinary cards is* $\frac{19}{17\cdot 49}$. Among five cards, only one rank can appear at least three times; pick it in 13 ways. When all four cards of this rank appear, there are 48 ways to pick the remaining card. When only three appear, there are four ways to pick the missing suit of this rank and $\binom{48}{2}$ ways to pick the other two cards. Hence there are $13\cdot 48\cdot(1+4\cdot 47/2)$ suitable sets of five cards. The desired probability is the ratio of this to $\binom{52}{5}$. Canceling factors in $\frac{13\cdot 48\cdot 95\cdot 120}{52\cdot 51\cdot 50\cdot 49\cdot 48}$ yields the claimed probability.

1.1.8. *From a standard* 52*-card deck, There are* $13^4\cdot 304$ *sets of six cards having at least one card in every suit.* We may have three cards in one suit and one in each other in $4\cdot\binom{13}{3}13^3$ ways. We may have two cards in each of two suits and one card in the other two in $\binom{4}{2}\binom{13}{2}^2 13^2$ ways. These are the only choices; we sum them.

1.1.9. *There are* $10\cdot 9\cdot 8\cdot 142$ *integers from* 0 *to* 99,999 *in which each digit appears at most twice (counting leading* 0*s as appearances.* Consider cases by how many different digits are used. There are $10_{(5)}$ integers using five digits. There are $10\binom{5}{2}9_{(3)}$ integers using four digits; first pick and place the repeated digit. When three digits are used, two are used twice; hence the number of integers of this type is $10\cdot 5\cdot 9\cdot\binom{4}{2}\cdot 8$. Summing the three cases yields the answer.

1.1.10. *There are* $11\binom{10}{4}\binom{6}{4}$ *distinguishable ways to order the letters of "Mississippi".* Choosing positions for the types of letters in stages, always the number of ways to do the next stage does not depend on how the previous stages were done. We place "M" in 11 ways, then choose four positions for "i" among the remaining 10 in $\binom{10}{4}$ ways, then choose four positions for "s" among the remaining 6 in $\binom{6}{4}$ ways, the put "p" in the remaining two positions. The rule of product then yields the answer.

1.1.11. *From four colors of marbles, there are* $\binom{15}{3}$ *distinguishable ways to have* 12 *marbles. There are* 4^{12} *ways to have* 12 *of the marbles in a row.* For distinguishable selections with repetition, we use the multiset formula: $\binom{12+4-1}{4-1}$. The number of ways to arrange a multiset depends on the number of elements of each type. However, when we put the elements in a row we are just making words: each position may have one of the four types, and all such words are distinguishable.

1.1.12. *If each New York City resident has a jar of 100 coins chosen from five types, then some two residents have equivalent jars.* The number of distinguishable jars of coins is the number of multisets of size 100 from five types. Using the formula $\binom{k+n-1}{n-1}$ for selections of k elements from n types, the value is $\binom{104}{4}$, which equals 4,598,126. Without being precise, cancelling factors yields $13\cdot 103\cdot 34\cdot 101$, which is clearly less than $5\cdot 10^6$. Since New York City has more than $7\cdot 10^6$ residents, the claim follows.

1.1.13. *When k is even, there are $2^{k/2-1}$ compositions of k with every part even (there are none when k is odd).* Halving each part yields a composition of $k/2$, and the map is reversible. There are 2^{n-1} compositions of n.

1.1.14. *Families of subsets.*

a) *There are $2^n - 2^{\lfloor n/2\rfloor}$ subsets of $[n]$ that contain at least one odd number.* There are 2^n subsets of $[n]$. Among these, $2^{\lfloor n/2\rfloor}$ subsets are restricted to the set of even numbers. The remainder have at least one odd number.

b) *There are $\binom{n-k+1}{k}$ k-element subsets of $[n]$ that have no two consecutive integers.*

Proof 1. When choosing k elements, the remaining $n-k$ must distribute among them to have at least one between each successive pair of chosen elements. Knowing how many go in each slot determines the k elements selected. Hence the legal choices correspond to solutions to $x_0 + x_1 + \cdots + x_k = n - k$ such that $x_1, \ldots, x_{k-1}$ are positive and x_0, x_k are nonnegative. Subtracting 1 from the variables required to be positive transforms these into nonnegative integer solutions of $y_0 + \cdots + y_k = n - 2k + 1$. By the selections with repetition model, the number of solutions is $\binom{n-2k+1+k+1-1}{k+1-1}$, which simplifies to $\binom{n-k+1}{k}$.

Proof 2. View the $n-k$ unchosen integers as dots in a row. We choose places for the selected integers between the dots (and on the ends), but avoidance of consecutive integers requires that no space is selected twice. We have $n-k+1$ allowable places and choose k of them for bars. The bars now mark the positions of k selected numbers.

c) *There are $n!$ choices of subsets $A_0, A_1 \ldots A_n$ of $[n]$ such that $A_0 \subset A_1 \subset \cdots \subset A_n$. There are $(n+2)^n$ choices such that $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n$.* When the sets have distinct sizes, we have $|A_i| = i$, since all the sizes are between 0 and n. Hence $A_0 = \varnothing$, and the elements of $[n]$ are added one by one in some order. The $n!$ possible orders correspond to the chains.

To determine a chain of the second type, it suffices to specify for each $x \in [n]$ the index i such that x first appears in the chain at A_i. Not appearing at all is also an option. Hence there are $n+2$ choices available for each x, and the choice made for x is not restricted by the choices made for other elements.

1.1.15. *The exponent on a prime p in the prime factorization of $\binom{2n}{n}$ is the number of powers p^k of p such that $\lfloor 2n/p^k \rfloor$ is odd.* We use the formula $\binom{2n}{n} = \frac{(2n)!}{n!n!}$. In $m!$, $\lfloor m/p \rfloor$ factors are divisible by p. In $\lfloor m/p^2 \rfloor$ of these, we have an extra factor of p. In $\lfloor m/p^3 \rfloor$, we have yet another factor of p, and so on. Hence the highest power of p that divides $m!$ is $\sum_{k\ge1}\lfloor m/p^k \rfloor$.

When $\lfloor 2n/p^k \rfloor$ is even, the number of multiples of p^k in $[2n]$ is twice the number in $[n]$; for example $\lfloor 10/2 \rfloor = 4$, and $\lfloor 5/2 \rfloor = 2$. When $\lfloor 2n/p^k \rfloor$ is odd, we get one extra: $\lfloor 6/2 \rfloor = 3$, but $\lfloor 3/2 \rfloor = 1$. The latter case occurs if and only if the remainder of n upon division by p^k is at least $p^k/2$.

Since the factors of p in the factorization of $n!$ are used (twice) to cancel factors of p in the factorization of $(2n)!$, we thus find that $(2n)!$ keeps an extra factor of p for each k where $\lfloor 2n/p^k \rfloor$ is odd.

Prime factors of $\binom{18}{9}$ and $\binom{20}{10}$. A prime p will be a factor if $\lfloor 2n/p^k \rfloor$ is odd for some k. We have $\lfloor 18/2 \rfloor = 9$ and $\lfloor 20/4 \rfloor = 5$, so 2 divides both. Since $\lfloor 18/3 \rfloor = \lfloor 20/3 \rfloor = 6$ and $\lfloor 18/9 \rfloor = \lfloor 20/9 \rfloor = 2$, 3 divides neither.

For higher primes, the squares are too big to give a nonzero contribution. We have $\lfloor 18/5 \rfloor = 3$ but $\lfloor 20/5 \rfloor = 4$, so 5 divides $\binom{18}{9}$ but not $\binom{20}{10}$. Since $\lfloor 18/7 \rfloor = \lfloor 20/7 \rfloor = 2$, 7 divides neither. However, 11, 13, and 17 yield 1 in each case, as does 19 in the latter case. Hence the prime divisors of $\binom{18}{9}$ are $\{2,5,11,13,17\}$, and those of $\binom{20}{10}$ are $\{2,11,13,17,19\}$.

1.1.16. *Given $v(a,b) = (\binom{a}{b-1},\binom{a}{b},\binom{a}{b+1})$, there do not exist distinct pairs (a,b) and (c,d) of positive integers such that $v(c,d)$ is a multiple of $v(a,b)$.* Suppose $v(c,d) = xv(a,b)$. We have $\binom{c}{d} = x\binom{a}{b}$, and then

$$\frac{d}{c-d+1}\binom{c}{d} = \binom{c}{d-1} = x\binom{a}{b-1} = x\frac{b}{a-b+1}\binom{a}{b} = \frac{b}{a-b+1}\binom{c}{d}$$
$$\frac{c-d}{d+1}\binom{c}{d} = \binom{c}{d+1} = x\binom{a}{b+1} = x\frac{a-b}{b+1}\binom{a}{b} = \frac{a-b}{b+1}\binom{c}{d}$$

Thus $(a-b+1)d = (c-d+1)b$ and $(b+1)(c-d) = (d+1)(a-b)$. The difference of these two equations yields $d-a+b = b-c+d$, and hence $a = c$. Now, since $\binom{a}{b+1} = \frac{a-b}{b+1}\binom{a}{b}$ and $\binom{c}{d+1} = \frac{c-d}{d+1}\binom{c}{d}$, and the ratios of $\binom{a}{b}$ to $\binom{c}{d}$ and $\binom{a}{b+1}$ to $\binom{c}{d+1}$ are the same, we have $\frac{a-b}{b+1} = \frac{c-d}{d+1}$. Thus $(d+1)(a-b) = (b+1)(c-d) = (b+1)(a-d)$. We obtain $(a+1)(d-b) = 0$, so also $d = b$.

1.1.17. *There are $\binom{m-1}{k-1}\binom{n+1}{k}$ lists of m 1s and n 0s having k runs of 1s.*

Proof 1 (case analysis). The number of runs of 0s may be $k-1$ (start and end with 1s), $k+1$ (start and end with 0s), or k (two cases, starting with 0s or with 1s). In each of these four cases, forming compositions of m and n with the right number of parts completely specifies the list.

For the 1s, we have $\binom{m-1}{k-1}$ compositions of m with k parts. For the 0s, the factor is $\binom{n-1}{k-2}$ or $\binom{n-1}{k}$ or $\binom{n-1}{k-1}$, the last in two cases. Summing these and applying Pascal's Formula three times yields $\binom{m-1}{k-1}\binom{n+1}{k}$.

Proof 2 (direct arguments). After forming a composition with k parts in $\binom{m-1}{k-1}$ ways for the 1s, these k *nonempty* runs are put into $n+1$ possible locations among the 0s (or at the ends). Runs of 1s go into distinct locations among the 0s, so there are $\binom{n+1}{k}$ ways to place them. [One can also place the 0s, with repetition allowed, among the runs of 1s, ensuring that the $k-1$ interior locations are nonempty. The number of ways is then the number of multisets of size $n-k+1$ from $k+1$ types.]

1.1.18. *Runs in subsets.*

a) *The number of subsets of* $[n]$ *with* k *runs is* $\binom{n+1}{2k}$. The runs in a subset correspond to runs of 1s in the incidence vector, separated by runs of 0s. We can specify the runs by inserting a bar before and after each run, separating it from the neighboring positions. Since there is at least one 0 between two runs of 1s in the incidence vector, the bars are placed in distinct positions. The allowable positions are between entries of the incidence vector, plus at the beginning or end. To specify k runs, we pick $2k$ of these $n+1$ positions, so the answer is $\binom{n+1}{2k}$.

Comment: There is also an analysis that considers cases depending on whether the first and/or last element is used or not. The cases yield binomial coefficients that combine to $\binom{n+1}{2k}$ by Pascal's Formula.

b) *The number of t-element subsets of* $[n]$ *with* k *runs is* $\binom{t-1}{k-1}\binom{n+1-t}{k}$. Determining the length of each run and the distances between runs determines the subset. Again consider the $2k$ bars specifying the runs; this time we must distribute t positions within the runs and $n-t$ outside them. By adding positions 0 and $n+1$ as extra positions before the first run and after the last, we guarantee $k+1$ nonempty bins outside the runs and have two composition problems. We need a composition of t with k parts to specify the lengths of the runs and a composition of $n+2-t$ with $k+1$ parts to specify the locations of the runs. There are $\binom{t-1}{k-1}$ of the former and $\binom{n+2-t-1}{k+1-1}$ of the latter, and we choose them independently.

c) *The number of t-element subsets of* $[n]$ *having exactly r_i runs of length s_i for $1 \le i \le m$, where $k = \sum_{i=1}^m r_i$ and $t = \sum_{i=1}^m r_i s_i$, is* $\frac{k!}{\prod_{i=1}^m r_i!}\binom{n+1-t}{k}$. Now we are given the lengths of the runs of 1s. To form the incidence vector, we permute them and position them. There are again k runs with total length t, so the factor $\binom{n+1-t}{k}$ for separating the runs of 1s remains. The runs can come in any order. However, all $r_i!$ ways of ordering the r_i runs of length i produce the same subset of $[n]$ (we assume that $s_1,\ldots,s_m$ are distinct). Thus there are $k!/\prod_{i=1}^m r_i!$ ways to order the runs of 1s.

1.1.19. *The number of binary strings of length n in which the number of copies of 00 is the same as the number of copies of 11 is 2 when $n = 1$ and is $2\binom{n-2}{\lceil n/2\rceil-1}$ when $n > 1$.* For $n = 1$, both strings are counted.

Consider $n > 1$. Let a and b be the number of 0s and number of 1s, respectively. If there are i runs of 0 and j runs of 1, then there are $a - i$ copies of 00 and $b - j$ copies of 11.

If the first and last bits differ, then $i = j$, and the desired condition holds if and only if $a = b$, which requires n even. If the first and last bits are 0, then $i = j + 1$, and we need $a = b + 1$, which requires n odd. Similarly, we need $a + 1 = b$ and n odd when the first and last bits are 1.

The needed property of the first and last bits holds in two ways. After ensuring this, the needed condition on a and b is satisfied in $\binom{n-2}{(n-2)/2}$ strings when n is even and in $\binom{n-2}{(n-1)/2}$ when n is odd. Thus the answer is $2\binom{n-2}{\lceil n/2\rceil-1}$ in both cases. The formula is not valid when $n = 1$ because the first and last bits are the same.

1.1.20. *The number of elements of $[3]^n$ with k odd entries having no 1 next to a 3 is $\sum_{j\ge 0}\binom{n-k+1}{j}\binom{k-1}{j-1}2^j$.* Let j be the number of runs of odd entries. Each run is all-1 or all-3, independently, since any two successive runs of odd entries are separated by at least one 2. With k odd entries in j runs, the run lengths of odd entries form a composition of k with j parts. Hence there are $\binom{k-1}{j-1}2^j$ ways to form the sublist of odd entries.

Altogether there are $n-k$ copies of 2. These are distributed into $j+1$ buckets, and all but the first and last must be nonempty (the list may or may not start or end with a 2). Hence there are $\binom{n-k+1}{j}$ ways to distribute the copies of 2.

Summing over the possibilities for j completes the proof.

1.1.21. *Inside a convex n-gon, $\binom{n}{4}$ pairs of chords cross.*

Proof 1 (brute force). Let a_n be the answer. Let $v_1, \ldots, v_n$ be the vertices in order. The vertices $v_1, \ldots, v_{n-1}$ form a convex $(n-1)$-gon, and within it a_{n-1} pairs of chords cross. To this we add the crossings involving chords at v_n. The chord from v_n to v_k crosses $(k-1)(n-k-1)$ other chords, so $a_n = a_{n-1} + \sum_{k=2}^{n-2}(k-1)(n-k-1)$. With $a_3 = 0$, we have $a_n = \sum_{r=4}^{n}\sum_{k=2}^{r-2}(k-1)(r-k-1)$. Proof by induction after guessing the answer from data, or application of identities from Section 1.2, may lead you to the answer $\binom{n}{4}$.

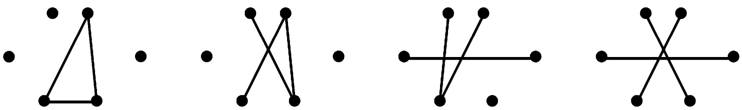
Proof 2 (combinatorial understanding). Each crossing involves two chords. Those two chords involve four endpoints. Thus every crossing corresponds to four points on the n-gon. Furthermore, each set of four points on the n-gon is the set of endpoints for exactly one pair of crossing chords. Hence the number of crossing pairs is $\binom{n}{4}$.

1.1.22. *If no three chords have a common internal point in the picture formed by drawing all $\binom{n}{2}$ chords of a convex n-gon, then the number of triangles is $\binom{n}{3}+4\binom{n}{4}+5\binom{n}{5}+\binom{n}{6}$.* We count the triangles according to how many corners lie on the boundary of the n-gon. A triangle with three boundary corners is determined by choosing three vertices of the n-gon.

A triangle with two boundary corners has a full chord as one side, and the other two sides extend to form full chords. The endpoints of these three chords are four points on the boundary. Hence such a triangle is associated with four vertices of the n-gon, chosen in $\binom{n}{4}$ ways. On the other hand, each choice of four vertices yields four such triangles.

A triangle with one boundary corner is determined by two chords from that point and one chord that crosses both of them. This leads to five vertices on the boundary. Each choice of five vertices determines five triangles in this way, so the number of triangles of this type is $5\binom{n}{5}$.

A triangle with no boundary corners is determined by three chords with no common endpoints, obtained by extending the sides. Thus six vertices must be chosen from the boundary to draw the chords. Each choice of six yields exactly one such triangle, with opposite pairs forming pairwise crossing chords.



Comment: Other ways to group and count the triangles produce more complicated formulas, which can be simplified to that above via identities. Having obtained a simple formula, one seeks a simple proof....

1.1.23. *Rolling dice.* Six dice each have three red faces, two green faces, and one blue face. The probability that three red faces, two green faces, and one blue face will show when all six are rolled is 5/36.

The six dice are objects; each shows some face. The number of arrangements of RRRGGB is $\binom{6}{3}\binom{3}{1}$, which equals 60. Each has probability $(\frac{3}{6})^3(\frac{2}{6})^2(\frac{1}{6})^1$ of occurring. Hence the desired probability is $60\cdot(27\cdot4\cdot1)/6^6$.

1.1.24. *In poker, a straight is more likely than a flush.* The number of sets of five cards from one suit is $4\binom{13}{5}$. For the number of sets of five cards with consecutive values, the lowest value can be any number from 1 to 10 (an ace can be considered high or low). Hence there are $10\cdot4^5$ such sets. After canceling common factors, the ratio of the number of straights to number of flushes simplifies to $\frac{10\cdot256}{13\cdot11\cdot9}$, which is about 1.989.

1.1.25. *The number of trapezoids defined by vertices of a regular n-gon is $n\binom{(n-1)/2}{2}$ if n is odd and $(n-3)\binom{n/2}{2}$ if n is even.* It suffices to count the pairs of parallel chords. Chords are parallel to a side or (when n is even) perpendicular to a diametric chord.

When n is odd, the number of chords parallel to a given side is $(n-1)/2$. Picking a side and picking two such chords yields the answer. The resulting trapezoids are distinct, because none are parallelograms since any two parallel chords have different lengths.

When n is even, the same analysis gives $\frac{n}{2}\binom{n/2}{2}$ pairs of chords parallel to sides. For any diametric chord, there are $(n-2)/2$ chords perpendicular to it, yielding $\frac{n}{2}\binom{(n-2)/2}{2}$ pairs of such chords. Every parallelogram has been counted twice. Each parallelogram is determined by having two specified corners among the first $n/2$ vertices, so there are $\binom{n/2}{2}$ paralleograms. Thus the number of trapezoids is $\frac{n}{2}\binom{n/2}{2} + \frac{n}{2}\binom{n/2-1}{2} - \binom{n/2}{2}$, which equals $(n-3)\binom{n/2}{2}$.

1.1.26. *The largest displacement $d(\pi)$ of a permutation of $[n]$ is $\lfloor n^2/2 \rfloor$, where $d(\pi) = \sum_{i=1}^{n} |i-\pi(i)|$.* Define π' from π by switching the elements in positions i and $i+1$. If these elements are both at most i or both at least $i+1$, then $d(\pi') = d(\pi)$. In the remaining case, one element is at most i and the other is at least $i+1$. Now the displacement is greater (by 2) in the permutation in which the larger of the entries in positions i and $i+1$ is in position i.

We conclude that if any two adjacent entries are in increasing order, then transposing them does not decrease the displacement. Hence the displacement is maximized by a permutation in which no two consecutive elements are in increasing order. The only such permutation is the reverse of the identity permutation. The displacement of this permutation is $\sum_{i=1}^{\lfloor n/2 \rfloor} 2(n-i)$, which equals $\lfloor n^2/2 \rfloor$.

1.1.27. *Bijection from the set A of permutations of $[n]$ to the set B of n-tuples $(b_1, \ldots, b_n)$ such that $1 \le b_i \le i$ for each i.* Each $a = a_1, \ldots, a_n \in A$ is a list of numbers. For each i, let b_i be the position of i in the sublist of a formed by the elements of $[i]$. Let $f(a)$ be the resulting list $b_1, \ldots, b_n$. By construction, $1 \le b_i \le i$, so $f(a) \in B$.

To prove that f is a bijection, we describe a function $g\colon B \to A$. We build $g(b)$ from an empty list by inserting numbers in the order $1, \ldots, n$. Before inserting i, the list consists of $\{1, \ldots, i-1\}$. We insert i to have position b_i. After processing b, we have a permutation of $[n]$.

To prove that f and g are bijections, it suffices to show that they are injective (in fact $g = f^{-1}$), since A and B are finite and have the same size. First consider f. Given distinct permutations in A, there is some least value j such that the subpermutations using elements $1, \ldots, j$ are different. Since they are the same earlier but differ at the jth step, the corresponding values of b_j are different.

For g, if two elements of B differ first at the jth index ($b_j \neq b'_j$), then the subpermutations of $1, \ldots, j$ in the two image permutations differ.

This bijection can also be described inductively.

1.1.28. *The number of exchanges of elements in a permutation needed to break all original adjacencies is $\lceil (n-1)/4 \rceil$, for $n \geq 6$.* Number to elements from 1 to n in order. Since two elements are moved, at most four original adjacencies can be broken by each exchange. There are $n-1$ original adjacencies; this proves the lower bound.

To achieve the bound, exchange $2i$ with $2\lceil (n-1)/4 \rceil + 2$, for $1 \leq i \leq r$, where $r = \lfloor (n-1)/4 \rfloor$. When $4 \mid (n-1)$, the first $2r$ even numbers are moved, while all odd numbers remain fixed, and all adjacencies are broken. When $4 \nmid (n-1)$, element $2r+2$ has been skipped and remains adjacent to $2r-1$ and $2r+1$. For the last switch, exchange $2r+2$ with n when $4 \mid n$, and exchange $2r+2$ with 1 when 4 divides $n-2$ or $n-3$.

1.1.29. *There are $(n!)^{2n}$ 0, 1-matrices with n^2 rows and n^2 columns such that (1) each row and column has exactly one* 1, *and (2) when the matrix is partitioned into n^2 blocks of n consecutive rows and n consecutive columns, each block contains exactly one* 1.

Choose the position of the 1 in each n-by-n tile successively, going across a row of tiles from left to right, processing rows in order from top to bottom. There are $(n-i+1)(n-j+1)$ choices available when dealing with the jth tile in the ith row, due to the $i-1$ tiles above it and the $j-1$ tiles to its left. In the ith row, the product of the numbers of choices is $(n-i+1)^n n!$, so over all rows we obtain $(m!)^n(n!)^m$.

More generally, when a square of size mn is divided into mn rectangular tiles of width m and height n (globally, m rows of n tiles), the same argument shows that the number of permutation matrics of order mn having exactly one 1 in each tile is $(m!)^n(n!)^m$.

1.1.30. *There are 2^{n-1} permutations π of $[n]$ such that $\pi(i+1) \leq \pi(i)+1$ for $1 \leq i \leq n-1$.* We view permutations as words. Call such a permutation *good*. Listing the possibilities for small n suggests the answer 2^{n-1}.

Proof 1 (induction on n). There is one good permutation of $[1]$. For $n > 1$, note that the constraint on the value following value n always holds. Hence if $\pi(1) = n$, then we can ignore n, and the remaining constraints are precisely those for a permutation of $[n-1]$. Hence there are 2^{n-2} good permutations starting with n.

If not at the beginning, then n must immediately follow element $n-1$; following any other would violate a constraint. Deleting n yields a good permutation of $[n-1]$, since the element now following $n-1$ (if any) is less than $n-1$. On the other hand, n can be inserted immediately following $n-1$ in any good permutation of $[n-1]$ to form such a good permutation of $[n]$. Both of these maps are injective, so the number of good permutations of $[n]$ in which n is not at the beginning is 2^{n-2}.

Combining the cases yields 2^{n-1}.

Proof 2 (combinatorial argument). We map good permutations into subsets of $[n-1]$. Given a good permutation π, let $f(\pi) = \{i\colon \pi(i) > \pi(i+1)\}$. To show that f is a bijection, we show that for each $S \subseteq [n-1]$, there is a unique permutation π of $[n]$ such that $f(\pi) = S$.

In a good permutation, the runs of increasing steps are consecutive numbers. Furthermore, the element after the end of a run must be smaller than the element starting it. Thus the elements of each run are smaller than the elements of each preceding run. Hence knowing the boundaries of the runs determines the permutation. For example, if the first run has k elements, then the permutation must start $n-k+1, n-k+2, \ldots, n$, and the next element will be just small enough to allow the next run to end at $n-k$.

Thus the set S of locations of descents determines exactly one good permutation. This is indeed the permutation π such that $f(\pi) = S$.

1.1.31. *If the set of elements in even-indexed positions of a graceful permutation of* $[2n]$ *is* $[n]$*, then the first and last elements differ by n,* where a permutation is *graceful* if the absolute differences between successive elements are distinct.

Call $1, \ldots, n$ “small” and $n+1, \ldots, 2n$ “large”. For a graceful permutation, the differences between neighboring elements sum to $\binom{2n}{2}$, which equals $2n^2 - n$.

When the small numbers occupy the even positions, each absolute difference is a large number minus a small number. Each number appears in two differences, except that the first number x and last number y only appear once. Hence the differences sum to $2X - x - (2Y - y)$, where X and Y are the sums of the large numbers and the small numbers, respectively. Since $X - Y = n^2$, we have $2n^2 - n = 2n^2 - (x - y)$; hence $x - y = n$.

Comment: The converse also holds. Suppose that $b_1 = b_{2n} + n$. In computing $T = \sum_{i=2}^{2n} |b_i - b_{i-1}|$ as a sum of positive differences, each b_i for $2 \le i \le 2n-1$ is weighted by $\delta_i \in \{-2, 0, +2\}$. We extend this to b_1 and b_{2n} by adding $(b_1 - b_{2n}) - n = 0$. We further observe $\sum_{i=1}^{2n} \delta_i = 0$, so

$$T = (\sum_{i=1}^{2n} \delta_i b_i) - n = [\sum_{i=1}^{2n} \delta_i (b_i - n)] - n$$

$$\le 2\sum_{i=1}^{n} (g_i - n) - 2\sum_{i=1}^{n} (s_i - n) - n = 2G - 2S - n = 2n^2 - n = T.$$

Thus equality holds throughout. In particular, if b_i is small, then $\delta_i = -2$. It follows that none of the terms $|b_{i+1} - b_i|$ has the form $|s_{j+1} - s_j|$. Hence the terms alternate between small and large. Since b_{2n} is small, the result follows.

1.1.32. *Counting necklaces.*

a) $(n-1)!/2$ *necklaces with n beads can be made from n distinct beads, for* $n \geq 3$. Starting from a given point, there are $n!$ ways to list the beads. Each necklace corresponds to $2n$ such listings, since we can start the list at any bead and go in either direction without changing the necklace.

b) $\frac{k^n - k}{n} + k$ *crowns with n beads can be made from k types of beads when n is prime.* Starting from a given point, there are k^n ways to list beads forming a circular pattern. A circular pattern arises n times in this way unless some string repeats with period less than n. For example, 111111000 would yield a circular pattern that arises nine times, while 110110110 would yield a circular pattern that arises only three times.

However, the length of the repeating string must divide n. Since n is prime, the only divisors are 1 and n. The k circular patterns made using only one bead arises only once among the k^n lists. The other lists all group into classes of size n; each giving one circular pattern.

1.1.33. *If a polynomial p in k variables is* 0 *at all points in* $\prod_{i=1}^{k} S_i$, *where* $|S_i| = d_i + 1$ *and p has degree* d_i *in variable* x_i, *for* $1 \leq i \leq k$, *then p is identically* 0. The base case $k = 1$ is the given hypothesis. Now consider $k > 1$. Fixing any choice $(x_1, \ldots, x_{k-1}) \in \prod_{i=1}^{k-1} S_i$ defines a polynomial in the one variable x_k. By hypothesis, its value is 0 for $x_k \in S_k$. By the case $k = 1$, its value is 0 everywhere. Now any value of x_k, not necessarily in S_k, defines a polynomial q in the variables $x_1, \ldots, x_{k-1}$ that is 0 when $(x_1, \ldots, x_{k-1}) \in \prod_{i=1}^{k-1} S_i$. By the induction hypothesis, this polynomial is 0 everywhere. Hence the original polynomial p is 0 everywhere.

1.1.34. *Combinatorial proof of* $(x+y)_{(n)} = \sum_k \binom{n}{k} x_{(k)} y_{(n-k)}$. When z is an integer, the falling factorial $z_{(n)}$ counts the simple n-words from an alphabet Z of size z. When Z is the disjoint union of an x-set X and a y-set Y, the words can also be formed by first choosing positions among the n positions in which to use letters from X. When there are k such positions, there are $x_{(k)}$ ways to fill them with a simple k-word from X, and each can be paired with any simple $(n-k)$-word from Y to form a simple n-word from Z. Summing over k counts each simple n-word from Z exactly once.

The Polynomial Principle extends to any number of variables, by induction on the number of variables (keep all but one variable fixed). Thus equality of two polynomials (in two variables) at all positive integer arguments implies equality as polynomials (and at all real arguments).

1.1.35. *Flags on poles.*

a) *There are $r^{(m)}$ ways to put m distinct flags on r flagpoles in a row.*

Proof 1. Place flag 1, then flag 2, etc. Each placement of a flag effectively splits its location into two locations, since later flags may go above or below it. With the number of choices iteratively rising, there are $r(r+1)\cdots(r+m-1)$ ways to complete the full process.

Proof 2. We obtain a permutation of the flags by listing in order the flags from the first pole, then the second, and so on. Each permutation can be associated with any nonnegative integer solution to $x_1+\cdots+x_r = m$ to specify how many flags go on each pole; the resulting arrangements are all distinct. Hence the answer is $m!\binom{m+r-1}{r-1}$, simplifying to $r^{(m)}$. The permutation and the distribution amounts for the poles can be chosen in either order, yielding the same computation.

b) *The real number identity $(x+y)^{(n)} = \sum_k \binom{n}{k} x^{(k)} y^{(n-k)}$.* When x and y are nonnegative integers, the left side counts the arrangements of n flags onto $x+y$ flagpoles. To count the same set in pieces, let k be the number of flags placed on the first x flagpoles. We can choose these flags in $\binom{n}{k}$ ways and then place these flags on the first x poles in $x^{(k)}$ ways and the remaining flags on the remaining poles in $y^{(n-k)}$ ways. Since each arrangement has some number of flags on the first x poles, each arrangement is counted exactly once when we sum over k.

Hence the identity holds for infinitely many choices of both x and y. By the Polynomial Principle, it holds for all real numbers x and y.

1.1.36. *There are $(n-1)!(2^n-1)$ ways to arrange n distinct flags on nonempty flagpoles in a rotating circle.*

Proof 1. Writing the flags in order from each flagpole yields a "circular permutation", listing $[n]$ in a circle. Since there are n possible starting points for writing down a circular permutation as a linear permutation, there are $(n-1)!$ circular permutations.

Any position in the circular permutation can be the last flag on a pole; we obtain the arrangements on poles by choosing any subset of the n flags to be the last flags on their poles. Since we choose each position in the circular permutation at most once, the poles we use are all nonempty. The number of poles is the number of positions chosen. There is no constraint on the number of positions chosen, except that we must choose at least one, because the flags must be placed.

We have shown that the arrangements correspond to a circular permutation of $[n]$ and a nonempty subset of the n flags; the product rule now completes the proof.

Proof 2. One can also apply circularity after placing the poles. There are $n!$ ways to write all the flags in order. There are $\binom{n-1}{r-1}$ ways to choose breakpoints to put these onto r poles, including the last position. Since rotating the circle does not change the arrangement, each circular arrangement using r poles arises in r ways by this procedure. Summing over r to count them all and applying the Committee-Chair Identity and the Binomial Theorem yields

$$\sum_r \frac{n!\binom{n-1}{r-1}}{r} = (n-1)!\sum_{r=1}^{n}\frac{n}{r}\binom{n-1}{r-1} = (n-1)!\sum_{r=1}^{n}\binom{n}{r} = (n-1)!(2^n-1).$$

1.1.37. *When p is prime, $\binom{n+p-1}{p}-\binom{n}{p}$ is divisible by n, for all n.* The quantity $\binom{n+p-1}{p}$ is the number of multisets of size p from n types of elements. Since $\binom{n}{p}$ is the number of p-subsets of $[n]$, the difference is the number of multisets in which some element is repeated. Group these multisets into groups of size n as follows; two multisets A and B are in the same group if B can be obtained from A by adding a constant to each element and reducing the values modulo n. Because p is prime, iteratively adding 1 to each element cannot repeat the multiset until n steps have been taken (that is, the pattern of multiplicities has no shorter period), so all the groups have size n.

The reason for subtracting $\binom{n}{p}$ is that this quantity is not divisible by n when p divides n. When p distinct elements are equally spaced modulo n, the grouping described above yields one group with only n/p sets. In that case $\binom{n+p-1}{p}$ differs from a multiple of n by n/p.

1.1.38. *The probability that a spinner with equally likely outcomes $1,\ldots,n$ sums to n in three spins is $\frac{(n-1)(n-2)}{2n^3}$.* There are n^3 equally likely outcomes of the experiment. The number of outcomes with sum n is the number of compositions of n with three parts. The number of these is $\binom{n-1}{2}$.

1.1.39. *Both sides of the identity below count the same set of ternary lists.*

$$\sum_{s=k}^{\lfloor n/2\rfloor}\binom{n+1}{2s+1}\binom{s}{k} = \binom{n-k}{k}2^{n-2k}$$

Both sides count the ternary $(n+1)$-tuples having a 2 in exactly k positions such that the copies of 2 separate the copies of 1 into $k+1$ portions of odd length.

On the left, start with $n+1$ positions, and chose an odd number (at least $2k+1$) to be nonzero. Chose k of the positions that are even-indexed relative to this sublist to receive 2. Between any two such positions, the number of copies of 1 is odd, and the number at the beginning or end is also odd.

On the right, begin with any ternary list of length $n-k$ having exactly k copies of 2. There are $\binom{n-k}{k}2^{n-2k}$ such lists; copies of 2 may be consecutive. Now insert one position immediately before each 2 and at the end. This position receives 1 or 0 as needed so that the number of copies of 1 in that portion between copies of 2 is odd. This choice is unique, so we obtain exactly one of the desired lists for each of the $\binom{n-k}{k}2^{n-2k}$ original lists of length $n-k$.

1.1.40. *Compositions of integers.*

a) *There are $\binom{k}{n}$ solutions in positive integers to $\sum_{i=1}^{n} x_i \le k$.* There are $\binom{r-1}{n-1}$ solutions in positive integers to $\sum_{i=1}^{n} x_i = r$; summing over r with $0 \le r \le k$ and applying the Summation Identity yields the answer $\binom{k}{n}$.

Directly, the solutions are determined by choosing n spaces to mark the partial sums, from the k spaces following k dots in a row. The value x_i corresponds to the distance from the $(i-1)$th chosen space to the ith, where by convention the 0th chosen space is before the first dot.

Another direct proof puts the desired solutions in bijective correspondence with the solutions to $\sum_{i=1}^{n+1} x_i = k+1$, by adding a positive variable x_{n+1} representing the slack in the inequality. These solutions correspond to the compositions of $k+1$ with $n+1$ parts; the number of them is $\binom{k+1-1}{n+1-1}$.

b) *There are 2^{k-1} compositions of k.* There are $\binom{k-1}{n-1}$ compositions of k with n parts. Sum over n and apply the Binomial Theorem.

Bijective proof: Group dots to build a composition. From a row of k indistinguishable dots, the first dot goes into the first part. Each subsequent dot can start a new part or enlarge the current part. Thus compositions are formed by making binary choices for $k-1$ dots, and each $(k-1)$-tuple of choices arises from exactly one composition of k.

c) *For $k > 1$, there are equally many compositions of k with an even number of parts and with an odd number of parts.* A composition is determined by choosing a subset S of the spaces among k dots; the resulting number of parts is $|S|+1$. When $k > 1$, half the subsets of a set of size $k-1$ have each parity (toggle the presence of the last element).

Comment: There are many natural bijections from A to B, where A and B are the sets of compositions of k having an even number and an odd number of parts, respectively. Essentially, they pair up odd and even subsets of the spaces between dots. For example, consider the map that combines the last two parts if the last part is 1 and splits 1 off the last part to form a new last part if the original last part exceeds 1.

d) *For $k \ge 2$, the number of compositions of k with an even number of even parts equals the number of compositions of k with an odd number of even parts.* Let A and B be the sets of compositions of k with an even number of even parts and an odd number of even parts, respectively. Define

$f: A \to B$ as follows. For $x \in A$, consider the first part, p. If $p = 1$, combine p with the second part. If $p > 1$, split off 1 from p to make a new first part. The 1 that appears or disappears does not affect the number of even parts. The other changed part changes by 1, so its parity changes. Hence the number of even parts changes by 1, which changes its parity.

Note that the first part in $f(x)$ is 1 if and only if the first part in x is not 1. Continuing through the other parts shows that the first difference between elements x and y of A causes a difference in $f(x)$ and $f(y)$. Hence f is injective. By the same argument, the function $g: B \to A$ defined in the same way is also injective. Hence $|A| = |B|$.

1.1.41. *Compositions of integers.*

a) *Over all compositions of k, the total number of parts is $(k+1)2^{k-2}$.* The compositions correspond to the subsets of the $k-1$ spaces in a row of k dots. Each j-element subset yields a composition with $j+1$ parts. The first dot starts a part in each composition. Each remaining dot starts a part in half of the compositions. Since the number of compositions is 2^{k-1}, the total number of parts is $2^{k-1} + (k-1)2^{k-2}$. (The same answer can be obtained by computing $\sum_{j=0}^{k-1}(j+1)\binom{k-1}{j-1}$ using techniques or identities from Section 1.2.)

Proof 1 (summation). Since there are $\binom{k-1}{j-1}$ compositions with j parts, the total equals $\sum_{j=1}^{k} j\binom{k-1}{j-1}$. By summing the Committee-Chair Identity over the committee size, we obtain $\sum_{j=1}^{k} j\binom{k-1}{j-1} = \sum_{j=1}^{k-1}\binom{k-1}{j-1} + \sum_{j=1}^{k-1}(j-1)\binom{k-1}{j-1} = 2^{k-1} + (k-1)2^{k-2} = (k+1)2^{k-2}$.

Proof 2 (bijection). Alternatively, compositions correspond to subsets of the spaces among k dots.

b) *Over all the compositions of k, there are $(k-m+3)2^{k-m-2}$ parts equal to m, where $1 \le m < k$.* Elements of the set $A_{k,m}$ being counted are expressible as the pairs (C, j), where C is a composition of k and j is a marked copy of m in C. Subtracting 1 from the marked copy of m yields a pair (C', j') in $A_{k-1,m-1}$. The map is injective and surjective, so the sets have the same size, as desired. This leaves the problem of counting $A_{k,1}$.

Proof 1 (direct argument using part (a)). Skipping any 1 in a composition of k leaves a composition of $k-1$, and each composition of $k-1$ with j parts arises in $j+1$ ways by doing this. That is, each composition of $k-1$ with j parts yields $j+1$ parts of size 1 among compositions of k. The answer is thus 1 for each composition of $k-1$ plus the answer of part (a) for $k-1$: $2^{k-2} + k2^{k-3} = (k+2)2^{k-3}$. The result is the special case of the claimed formula for $m = 1$.

Proof 2 (induction on k). Let $a_k = |A_{k-1}|$. Note that $a_2 = 2 = (2-1+3)2^{2-1-2} = 2$. For $k > 2$, group the compositions by the last part.

With last part 1, the total number of 1s is $a_{k-1}+2^{k-2}$, since there are 2^{k-2} compositions of $k-1$. With last part j, where $2 \le j \le k-2$, the total is a_{k-j}, and there is 1 with last part $k-1$. Thus $a_k = 1 + 2^{k-2} + \sum_{j=1}^{k-2} a_{k-j}$. We can apply the induction hypothesis and then perform the sum. To avoid performing the sum, subtract consecutive instances to obtain for $k \ge 3$ that $a_k - a_{k-1} = 2^{k-2} - 2^{k-3} + a_{k-1}$. Now the induction hypothesis yields $a_k = 2(k+1)2^{k-4} + 2^{k-3} = (k+2)2^{k-3}$.

Proof 3 (induction variation). For $k > 2$, reduce the last part by 1 to obtain a composition of $k-1$. Each such composition arises twice, by deleting a final 1 or by reducing a final part larger than 1. Counting 1s over all resulting compositions thus yields $2a_{k-1}$, but we lost one for each of the 2^{k-2} compositions of k ending in 1, and we gained one for each of the 2^{k-3} compositions of k ending in 2. Hence $a_k = 2a_{k-1} + 2^{k-2} - 2^{k-3} = (k+2)2^{k-3}$.

Proof 4 (overall induction, sketch). One can also do the whole problem by induction on k for fixed m. The inductive idea is like that in Proof 2 or Proof 3, with adjustments for the copies of m that arise or disappear. The computations are not as clean as above, so we omit them.

c) $1 + \sum_{m=1}^{k-1}(k-m+3)2^{k-m-2} = (k+1)2^{k-2}$. By parts (a) and (b), both sides count all parts in all compositions of k (the extra 1 counts the composition whose only part is k).

1.1.42. *The Weights Problem. The set $S_k = \{1, 3, \dots, 3^{k-1}\}$ permits the checking of all integer weights from 1 through $(3^k-1)/2$ on a balance scale, and no other choice of k known weights permits more values to be checked.*

Let $f(k) = (3^k-1)/2$. We first prove that S_k permits us to balance object A_n of integer weight n for $1 \le n \le f(k)$. It suffices to express n as $\sum_{i=0}^{k-1} b_i 3^i$, where each $b_i \in \{-1, 0, 1\}$, because then interpreting $-1, 0, 1$ for b_i to mean "same side as A_n", "off the balance", and "side opposite to A_n" yields an explicit configuration of the weights that balances A_n.

We find the desired numbers $\{b_i\}$ using the ternary expansion of the number $n' = n + f(k)$. The equation $n = \sum_{i=0}^{k-1} b_i 3^i$ holds if and only if the equation $n' = \sum_{i=0}^{k-1}(b_i+1)3^i$ holds, because the geometric sum yields $(3^k-1)/2 = \sum_{i=0}^{k-1} 3^i$. Since $n \le f(k)$, we have $n' \le 2f(k) = 3^k-1$. Ternary expansion guarantees a (unique) expression of n' as $n' = \sum_{i=0}^{k-1} a_i 3^i$ with each $a_i \in \{0, 1, 2\}$. Setting $b_i = a_i - 1$ yields an explicit way to weigh n.

We also must prove that no other set of weights can balance more values. We count the possible configurations: each weight can be placed on the left, on the right, or omitted, generating 3^k possible configurations. The configuration that omits all weights balances no nonzero weight. Of the remaining 3^k-1 configurations, each balances the same weight as the configuration obtained by switching the left pan and right pan. Hence at most $(3^k - 1)/2$ distinct values can be weighed.

The construction for the lower bound can also be established using induction on k. The advantage of the bijective proof is that it gives an explicit description of the configuration used to balance a given weight.

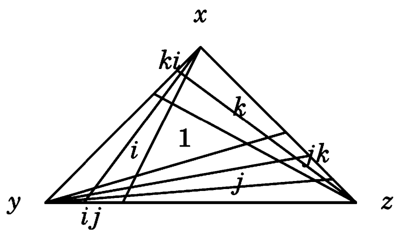
1.1.43. *Using weights $w_1 \le \cdots \le w_n$ on a two-pan balance, where $S_j = \sum_{i=1}^{j} w_i$, every integer weight from 1 to S_n can be weighed if and only if $w_1 = 1$ and $w_{j+1} \le 2S_j + 1$ for $1 \le j < n$.* For sufficiency, we use induction on n. When $n = 1$, the condition forces $w_1 = 1$, and the weight 1 can be balanced. For the induction step, consider $n > 1$, and suppose that the condition is sufficient for $n - 1$ weights. For $1 \le i \le S - w_n$, the induction hypothesis implies that we can weigh i using $\{w_1, \ldots, w_{n-1}\}$. With w_n also available, we can also weigh $w_n - i$ and $w_n + i$, so we can weigh every weight from $w_n - S_{n-1}$ to $w_n + S_{n-1} = S_n$ using $\{w_1, \ldots, w_n\}$. Since $w_n - S_{n-1} \le S_{n-1} + 1$ by hypothesis, we can weigh every weight up to S_n.

For necessity, suppose we can balance all weights from 1 to S_n. The second largest possibility is $S_n - w_1$, required to be $S_n - 1$, so $w_1 = 1$. If $w_{j+1} > 2S_j + 1$ for some j, then let $W = S_n - 2S_j - 1$; we claim that W cannot be weighed. The largest weight achievable without putting all of $\{w_{j+1}, \ldots, w_n\}$ in one pan is $S_n - w_{j+1} < W$, but the smallest weight achievable using all of $\{w_{j+1}, \ldots, w_n\}$ in one pan is $S_n - 2S_j$, which exceeds W.

1.1.44. *Regions in cevian arrangements.* From the three points x, y, z on a circle, chords emerge and reach the circle between the other two points. When counting regions, we can view this as chords from a vertex of a triangle to the opposite side (these are called *cevians* in geometry).

From each cevian arrangement with i, j, k chords emerging from x, y, z, respectively, we can reach every other such arrangement by iteratively sliding the foot of one chord. When a chord reaches the intersection of two other chords, we temporarily lose a region, but the count is restored when the chord emerges from the intersection. Thus every configuration without triple intersection points achieves the maximum number of regions.

Proof 1. The regions become easy to count when the chords from each vertex reach the circle near the next vertex, cyclically. The chords from x and y then form a small grid of ij regions near y. Near the xy edge, the chords from x form i regions. Repeating this cyclically counts all the regions except one central region, and the total is $ij + jk + ki + i + j + k + 1$.



Proof 2. Having observed that the maximum occurs when there are no triple-points and that each chord intersects every other, one can count the regions formed as chords are placed. First chords from x each split 1 region when added. Then the chords from y split $i+1$ regions each. Finally the chords from z split $i+j+1$ regions each. Starting from a single initial region, the total becomes

$$1 + i(1) + j(i+1) + k(i+j+1) = ij + ik + jk + i + j + k + 1.$$

Proof 3. Having observed that the maximum occurs when there are no triple-points and that each chord intersects every other, one can treat the configuration as a planar graph and apply Euler's Formula. Here the computations are a bit more complicated, and we are not assuming Euler's Formula.

1.1.45. *When n is divisible by r, and a k-set is chosen from $[n]$ uniform at random, where* $\gcd(k,r)=1$*, the probability that the sum of the k-set is divisible by r is* $1/r$. We prove more generally that the sum is equally distributed over the congruence classes modulo r. For any k-set, adding 1 to each element (n turns into 1) produces another k-set whose sum modulo r is larger by k. Since $\gcd(k,r)=1$, the original congruence class is not revisited until r steps later, after one k-set has been found in each congruence class. Since n is divisible by r, the translates of a given k-set contribute equally to the r congruence classes. Since translation partitions the k-sets into disjoint classes, and the claim holds for each class, the distribution over all the k-sets is also uniform.

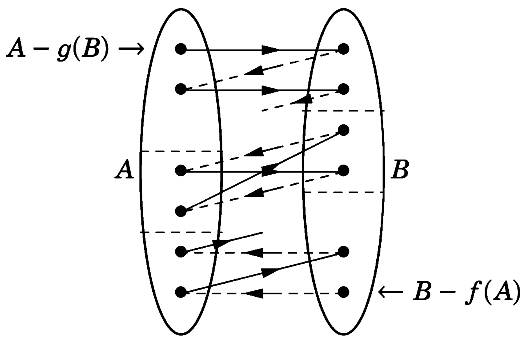
1.1.46. *For $n,m,k\in\mathbb{N}$ with $k\le n$,* $\sum_{j=0}^{m}\frac{k\binom{n}{k}\binom{m}{j}}{(k+j)\binom{n+m}{k+j}}=1$. Consider a deck of n blue cards and m red cards. A player pulls cards at random without replacement and wins when k blue cards are obtained. The probability of winning is 1, since $k\le n$. We compute the probability that the player wins after drawing exactly j red cards, where $0\le j\le m$. The probability that exactly k of the first $k+j$ cards are blue is $\binom{n}{k}\binom{m}{j}/\binom{n+m}{k+j}$. The probability that the last card is blue given that exactly k of the first $k+j$

cards are blue is $k/(k+j)$. Hence the probability of winning after drawing exactly j red cards is $\frac{k\binom{n}{k}\binom{m}{j}}{(k+j)\binom{n+m}{k+j}}$. Summing over j completes the proof.

1.1.47. *If $f\colon A \to B$ and $g\colon B \to A$ are injections, then there exists a bijection $h\colon A \to B$, and hence A and B have the same cardinality.* (Schroeder-Bernstein Theorem)

We view A and B as disjoint sets, making two copies of common elements. For each element z of $A \cup B$, we define the *successor* of z to be $f(z)$ if $z \in A$, and $g(z)$ if $z \in B$. The *descendants* of z are the elements that can be reached by repeating the successor operation. We say that z is a *predecessor* of w if w is the successor of z. Because f and g are injective, every element of $A \cup B$ has at most one predecessor. The *ancestors* of z are the elements that can be reached by repeating the predecessor operation.

The *family* of z consists of z together with all its ancestors and descendants; call this $F(z)$. We use the structure of families to define a one-to-one correspondence between A and B. The successor operation defines a function f' on $A \cup B$; below we show several possibilities for families using a graphical description of f'.



First suppose that z is a descendant of z. Because every element has at most one predecessor, in this case $F(z)$ is finite (repeatedly composing the successor function leads to a "cycle" of elements involving z). Applying f' alternates between A and B, and thus $F(z)$ has even size. For every $x \in A$ in $F(z)$, we pair x with $f(x)$; because $F(z)$ has even size, this is a one-to-one correspondence between $F(z) \cap A$ and $F(z) \cap B$.

Otherwise, $F(z)$ is infinite. In this case, the set $S(z)$ of ancestors of z may be finite or infinite. When $S(z)$ is finite, it contains an *origin* that has no predecessor (all elements of $F(z)$ have the same origin). If $S(z)$ has an origin in B, then for every $x \in A \cap F(z)$ we pair x with its predecessor $g^{-1}(x)$; because B contains the origin, $g^{-1}(x)$ exists. When $S(z)$ is infinite or has an origin in A, we pair x with its successor $f(x)$.

Because every element has at most one predecessor, the pairing we have defined is a one-to-one correspondence between the elements of A and the elements of B within $F(z)$. Since the families are pairwise disjoint, it is also a one-to-one correspondence between A and B. In more technical language, we have defined the function $h\colon A \to B$ by $h(x) = g^{-1}(x)$ when the family of x has an origin in B, and $h(x) = f(x)$ otherwise. The function h is the desired bijection.

1.2. IDENTITIES

1.2.1. *Combinatorial proofs of* $\binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k}$ *and* $\binom{m+n}{m+k}\binom{m+k}{k} = \binom{m+n}{m}\binom{n}{k}$. To choose $k+1$ elements from $[n]$, one can first choose k and then choose one element from the remaining $n-k$. This marks one element as special, which could be any of the $k+1$ chosen elements, so each $(k+1)$-set has been counted $k+1$ times.

For the second equality, both sides are 0 unless $0 \le k \le n$. Both sides count the ternary lists of length $m+n$ in which m positions are 0 and k of the remaining n positions are 1. On the right side, choose the positions for 0s and then the positions for 1s. On the left side, first choose all the positions for 0s and 1s, and then among them choose the positions for 1s.

1.2.2. $\sum_{k=0}^{n}\binom{m+k}{k} = \binom{m+n+1}{m+1}$. After applying complementation to convert the summand to $\binom{m+k}{m}$, the Summation Identity evaluates the sum.

1.2.3. $\binom{n}{k} - \binom{n-2}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1}$. Using Pascal's Formula twice,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-2}{k}.$$

1.2.4. *Pascal's Formula holds for the extended binomial coefficient.* We take the coefficient $\binom{u}{k}$ to be 0 when $k < 0$, so the formula holds when $k = 1$. For $k > 1$, we use $\binom{v}{k} = \frac{1}{k!}\prod_{i=0}^{k-1}(v-i) = \frac{v-k+1}{k}\binom{v}{k-1}$ with v equal to $u-1$ and then u to compute

$$\binom{u-1}{k} + \binom{u-1}{k-1} = \frac{u-k}{k}\binom{u-1}{k-1} + \binom{u-1}{k-1} = \frac{u}{k}\binom{u-1}{k-1} = \binom{u}{k}.$$

1.2.5. $\sum_k \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n}$. Setting $l = n-k$, the sum becomes $\sum_l \binom{r}{m+n-l}\binom{s}{l}$, and the value is given immediately by the Vandermonde convolution.

1.2.6. $\sum_{k=0}^{n}\binom{m+k-1}{k}=\sum_{k=0}^{m}\binom{n+k-1}{k}$ *and* $\sum_k\binom{n}{k}\binom{m}{r+k}=\binom{n+m}{n+r}$. For the first equality, applying complementation to convert the summands to $\binom{m+k-1}{m-1}$ and $\binom{n+k-1}{n-1}$ allows the Summation Identity to evaluate the two sides to $\binom{m+n}{m}$ and $\binom{m+n}{n}$, which are equal.

Applying complementation to the second factor in the summand converts the sum to $\sum_k\binom{n}{k}\binom{m}{m-r-k}$, which evaluates by the Vandermonde convolution to $\binom{n+m}{m-r}$, which equals $\binom{n+m}{n+r}$.

1.2.7. *Evaluation of* $\binom{-1}{k}$. Using the definition,

$$\binom{-1}{k}=\frac{1}{k!}\prod_{i=0}^{k-1}(-1-i)=\frac{(-1)^k}{k!}\prod_{i=0}^{k-1}(i+1)=(-1)^k.$$

1.2.8. *Summing the first n positive numbers.* In the formulas $i^2=2\binom{i}{2}+\binom{i}{1}$ and $i^3=6\binom{i}{3}+6\binom{i}{2}+\binom{i}{1}$, the left side counts k-tuples from an i-set, where $k\in\{2,3\}$. On the right for $k=2$, we can choose two distinct elements in $\binom{i}{2}$ ways and list them in either order, while if we use the same element twice there are i choices. For $k=3$, there are six orders in which we can list three distinct elements. When using only two elements, we choose them, pick which of the two to use only once, and pick its location, yielding $6\binom{i}{2}$. Again there are i ways to use only one element.

We use the Summation Identity to perform the sums.

$$\sum_{i=1}^{n}i=\sum_{i=1}^{n}\binom{i}{1}=\binom{n+1}{2}=\frac{n(n+1)}{2}.$$

$$\sum_{i=1}^{n}i^2=\sum_{i=1}^{n}2\binom{i}{2}+\binom{i}{1}=2\binom{n+1}{3}+\binom{n+1}{2}=\frac{n(n+1)(2n-2+3)}{6}.$$

$$\sum_{i=1}^{n}i^3=\sum_{i=1}^{n}6\binom{i}{3}+6\binom{i}{2}+\binom{i}{1}=6\binom{n+1}{4}+6\binom{n+1}{3}+\binom{n+1}{2}$$
$$=n(n+1)\left(\frac{(n-1)(n-2)}{4}+(n-1)+\frac{1}{2}\right)=n(n+1)\frac{n^2+n}{4}.$$

As an application,

$$\sum_{i=1}^{n}(2i^3+3i^2-5i)=\frac{n^2(n+1)^2}{2}+\frac{n(n+1)(2n+1)}{2}-5\frac{n(n+1)}{2}$$
$$=n(n+1)\left(\frac{n(n+1)}{2}+\frac{2n+1}{2}-\frac{5}{2}\right)=\frac{1}{2}n(n+1)(n-1)(n+4).$$

1.2.9. *There are $d_{m,n}$ ways to put m white and n black marbles into boxes if each box has at most one marble of each color and no box is empty until all marbles are used.* The boxes in order correspond to steps in a Delannoy path. The steps can be horizontal (white marble) or vertical (black marble) or diagonal (one marble of each color). When all marbles are used, the path will reach (m, n).

1.2.10. *A triple product identity.* We use the Committee-Chair Identity twice, reorder the factors, and use it two more times.

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k-1}\frac{n}{k+1}\binom{n-1}{k}\frac{n+1}{k}\binom{n}{k-1}$$
$$= \binom{n-1}{k}\left[\frac{n+1}{k+1}\frac{n}{k}\binom{n-1}{k-1}\right]\binom{n}{k-1} = \binom{n-1}{k}\binom{n+1}{k+1}\binom{n}{k-1}.$$

1.2.11. *Identities by induction, using Pascal' Formula.*

a) *The binomial coefficient formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.* The formula holds for $n = 0$ under the convention that the "factorial" of a negative number is infinite. For $n > 1$, Pascal's Formula and the induction hypothesis yield $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} = \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{n-k}{n}\frac{n!}{k!(n-k)!} + \frac{k}{n}\frac{n!}{k!(n-k)!} = \frac{n!}{k!(n-k)!}$.

1.2.12. *When flipping 100 fair coins, the number of heads and tails are more likely to differ by 2 than be equal.*

Proof 1 (computation with factorials). The probability of equal numbers is $\binom{100}{50}/2^{100}$. The probability of differing by 2 is $2\binom{100}{49}/2^{100}$, since either heads or tails may be extra. For the ratio, we cancel like factors and compute $\frac{50!50!}{2\cdot 49!51!} = \frac{50}{2\cdot 49} < 1$.

Proof 2 (combinatorial argument). Given any string with 50 heads and 50 tails, switching the last flip yields a string in which the numbers differ by 2. Distinct strings get mapped to distinct strings, so the map is injective. Furthermore, strings in which the last entry is in the minority do not arise, since flipping it from equal weight makes its new value occur 51 times. Hence there are strictly more differing by 2.

b) *The Summation Identity $\sum_{i=0}^{n}\binom{i}{k} = \binom{n+1}{k+1}$ for $n, k \ge 0$.* For $n = 0$, the identity reduces to $\binom{0}{k} = \binom{1}{k+1}$; both sides equal 1 if $k = 0$ and 0 if $k > 0$. For $n > 1$, the induction hypothesis and Pascal's Formula yield $\sum_{i=0}^{n}\binom{i}{k} = \binom{n}{k} + \sum_{i=0}^{n-1}\binom{i}{k} = \binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$.

c) *The Binomial Theorem $(x + y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$.* For $n = 0$, we have $(x + y)^0 = 1 = \binom{0}{0}x^0y^0$. For $n > 0$, the induction hypothesis gives $(x + y)^{n-1} = \sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-1-k}$. We multiply both sides by $(x + y)$ and simplify the resulting expansion. To combine terms where the exponents agree on x and agree on y, we shift the index in the first summation. We then use Pascal's Formula to combine corresponding terms. For the extra terms, $\binom{n-1}{n-1} = 1 = \binom{n}{n}$ and $\binom{n-1}{0} = 1 = \binom{n}{0}$; these become the top and bottom terms of the desired summation. The full computation is

$$\begin{aligned}(x+y)^n &= (x+y)\sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-1-k}\\ &= \sum_{k=0}^{n-1}\binom{n-1}{k}x^{k+1}y^{n-1-k} + \sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-k}\\ &= \sum_{l=1}^{n}\binom{n-1}{l-1}x^l y^{n-l} + \sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-k}\\ &= x^n + \left(\sum_{k=1}^{n-1}\left[\binom{n-1}{k-1}+\binom{n-1}{k}\right]x^k y^{n-k}\right) + y^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}\end{aligned}$$

d) *An alternating sum:* $\sum_{i=0}^{k}(-1)^i\binom{n}{k-i} = \binom{n-1}{k}$. We prove this for $n, k \ge 0$. For $n = 0$, the conventions for binomial coefficients yield 0 on both sides unless $k = 0$ (where they equal 1). For $n > 0$, the induction hypothesis and Pascal's Formula yield

$$\begin{aligned}\sum_{i=0}^{k}(-1)^i\binom{n}{k-i} &= \sum_{i=0}^{k}(-1)^i\left[\binom{n-1}{k-i}+\binom{n-1}{k-i-1}\right]\\ &= \sum_{i=0}^{k}(-1)^i\binom{n-1}{k-i} + \sum_{i=0}^{k}(-1)^i\binom{n-1}{k-1-i}\\ &= \binom{n-2}{k}+\binom{n-2}{k-1} = \binom{n-1}{k}.\end{aligned}$$

1.2.13. *Combinatorial transformations for summing* $\min\{i,j\}$ *and* $\max\{i,j\}$.

a) $\sum_{i=1}^{n}\sum_{j=1}^{n}\min\{i,j\} = \sum_{k=1}^{n}k^2$.

Proof 1. Consider an arrangement of unit cubes piled atop the square with opposite corners $(0,0)$ and (n,n) in the plane. The pile of cubes in position (i,j) (that is, with upper right corner (i,j)) has height $\min\{i,j\}$. Thus the sum on the left counts the cubes. The number of positions where the pile has height at least $(n-k+1)$ is the number of pairs (i,j) such that both i and j belong to the set $\{n-k+1,\ldots,n\}$. There are k^2 such pairs, so grouping the cubes by the height of their position yields the sum $\sum_{k=1}^{n}k^2$. (Without geometry, this argument is the same as summing the entries of a matrix with $\min\{i,j\}$ in position (i,j).)

Proof 2. Both sums count the squares with positive integer side-lengths formed by the lines $y = 0,\ldots,y = n$ and $x = 0,\ldots,x = n$. There are k^2 such squares with side-length $n+1-k$, so the sum on the right counts them by size. The sum on the left groups them by the upper right corner. The number of squares in the set whose upper right corner is the point (i,j) is precisely $\min\{i,j\}$, and we sum over all choices for i and j.

Proof 3. Both sums count the 3-tuples of integers from $[n]$ in which the third element is smallest. When the first two elements are (i,j), there are $\min\{i,j\}$ choices for the third element. When the third element is r, there are $n-r+1$ choices for each of the first two elements. Letting $k=n-r+1$ again yields the sum $\sum_{k=1}^{n} k^2$.

b) $\sum_{k=1}^{n} k^2 = 2\binom{n+1}{3}+\binom{n+1}{2}$. Both sides count 3-tuples $(r,s,t)\in[n+1]^3$ such that $t>\max\{r,s\}$. The sum on the left counts the triples according to the value of t; when $t=k+1$, there are k^2 ways to specify r and s. The terms on the right group the triples according to whether $r=s$. If so, then we pick two elements and put the larger in t. If not, then we pick three, put the largest in t, and choose either order for r and s.

c) $\sum_{i=1}^{n}\sum_{j=1}^{n}\min\{i,j\} = \frac{1}{6}n(n+1)(2n+1)$ *and* $\sum_{i=1}^{n}\sum_{j=1}^{n}\max\{i,j\} = \frac{1}{6}n(n+1)(4n-1)$. For the first sum, we invoke parts (a) and (b) and then compute $2\binom{n+1}{3}+\binom{n+1}{2} = \frac{1}{3}(n+1)n(n-1)+\frac{1}{2}(n+1)n = \frac{1}{6}(n+1)n[2n-2+3]$. For the second, since $\min\{i,j\}+\max\{i,j\}=i+j$, we subtract the first sum from $\sum_{i=1}^{n}\sum_{j=1}^{n}(i+j)$, which equals $n\sum_{i=1}^{n} i + n\sum_{j=1}^{n} j$, or $2n\binom{n+1}{2}$. The final value is $(n+1)n[n-\frac{1}{6}(2n+1)]$, which yields the desired formula.

Comment: These identities can also be obtained by algebraic manipulation of known identities involving things like sums of squares.

1.2.14. $\sum_{j=1}^{m}(m-j)2^{j-1} = 2^m-m-1$.

Proof 1 (induction). Let $f(m)$ denote the given sum. Note that $f(m)-f(m-1)=\sum_{j=1}^{m-1}2^{j-1}=2^{m-1}-1$. Also $f(0)=0$, since the sum is empty. Therefore,

$j\}$.

$$f(m)=\sum_{i=1}^{m}(f(i)-f(i-1))=\sum_{i=1}^{m}(2^{i-1}-1)=2^m-1-m.$$

Proof 2 (counting two ways). The family of all subsets of $[m]$ with size at least two has size 2^m-1-m. The given sum counts this by the position of the next-to-last 1 in the incidence vector. When the next-to-last 1 is in position j, there are $m-j$ choices for the rightmost 1 and 2^{j-1} ways to fill the first $j-1$ positions. The sum counts precisely the incidence vectors with at least two 1s, because those are the vectors that have a next-to-last 1.

1.2.15. *Combinatorial arguments for identities.*

a) $\binom{2n}{n}=2\binom{2n-1}{n-1}$. To pick a set of size n from $[2n]$, one can use element n and choose $n-1$ from the remaining $2n-1$ elements to add, or omit element n and choose $n-1$ from the remaining $2n-1$ elements to omit.

b) $\sum_k \binom{k}{l}\binom{n}{k} = \binom{n}{l}2^{n-l}$. The kth term in the sum counts the ways to form a committee of size k with a subcommittee of size l from a set of n people, choosing the committee first and then the subcommittee. When we sum over k, we are considering all possible sizes for the committee, so the sum counts all possible committees with a subcommittee of size l.

Selecting the subcommittee first can be done in $\binom{n}{l}$ ways, and then the rest of the committee can be filled by choosing an arbitrary subset of the people that remain. Thus the right side also counts this set.

The proof can equivalently be phrased by saying that both sides count the ternary n-tuples with l zeros.

c) $\sum_{k=1}^{n} q^{k-1} = \frac{q^n-1}{q-1}$ *for* $q, n \in \mathbb{N}$. Consider a tournament with q^n players in which games involve q players and exactly one survives. In the first round, there are q^{n-1} games. In the jth round, q^{n-j+1} players remain and there are q^{n-j} games. In the nth round, there is one game, and the winning survives. Setting $k = n+1-j$ shows that the left side of the identity is the total number of games. On the other hand, $q^n - 1$ players must be eliminated, and each game eliminates $q-1$ players, so the right side also counts the games.

d) $\sum_{i=1}^{n} i(n-i) = \sum_{i=1}^{n} \binom{i}{2}$. Both sides count the 3-element subsets of $[n+1]$. The left side groups them according to the middle element; there are $i(n-i)$ triples in which the middle element is $i+1$. The right side groups them according to the top element; there are $\binom{i}{2}$ triples in which the top element is $i+1$.

1.2.16. *Strehl's Identity:* $\sum_k \binom{n}{k}^2\binom{2k}{n} = \sum_k \binom{n}{k}^3$.

Proof 1 (Vandermonde convolution). Special cases of the Vandermonde convolution include $\sum_k \binom{r}{k}\binom{s}{k} = \binom{r+s}{r}$ and $\sum_i \binom{r}{i}\binom{r}{n-i} = \binom{2r}{n}$. With these (at the beginning and end), the Subcommittee Identity, and reversing the sum on k, we compute

$$\begin{aligned}
\sum_k \binom{n}{k}^3 &= \sum_{i+j=n} \binom{n}{i}\binom{n}{j}\binom{i+j}{i} = \sum_{i+j=n} \binom{n}{i}\binom{n}{j}\sum_k \binom{i}{k}\binom{j}{k} \\
&= \sum_k \sum_{i+j=n} \binom{n}{i}\binom{i}{k}\binom{n}{j}\binom{j}{k} = \sum_k \binom{n}{k}^2 \sum_{i+j=n} \binom{n-k}{n-i}\binom{n-k}{n-j} \\
&= \sum_k \binom{n}{n-k}^2 \sum_{i+j=n} \binom{n-k}{j}\binom{n-k}{i} = \sum_k \binom{n}{k}^2 \sum_{i+j=n} \binom{k}{j}\binom{k}{i} \\
&= \sum_k \binom{n}{k}^2\binom{2k}{n}.
\end{aligned}$$

Proof 2 (counting two ways). Given n distinct red cards and n dis-