Complex Analysis for Mathematics and Engineering, 6th Edition

Instruction Manual

In this manual we provide three items:

- **Answers to Selected Exercises**
  Of course, these answers are not provided in the text. We frequently give detailed solutions so that you can make them for student use if you wish.
- **Sample Examination Questions**
  We list possible examination questions.
- **Sample Course Outline**
  We provide an outline for a one semester course consisting of 42 class days. In the outline we suggest routine homework assignments. The outline and accompanying assignments are representative of strategies that have worked in a variety of institutions. We also have research projects listed on the Jones and Bartlett website.

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Answers to Selected Exercises

Chapter 1

Section 1.1. The Origin of Complex Numbers: page 11

2. As the text indicates, when complex numbers arose as solutions to quadratic equations such as $x^2+1=0$ they were easily dismissed as meaningless. Bombelli showed, however, that complex numbers were indispensable in obtaining *real* solutions to certain cubic equations. For example, the depressed cubic $x^3-15x-4=0$ clearly has $x=4$ as a solution, but Bombelli's technique gets this number only by the calculation $(2+i)+(2-i)=4$. Thus, the utility of complex numbers in producing real solutions to some cubic equations could not be ignored.

4a. Following the hint let $b=0$ and $c=3$. The two points labeled as E and F in Figure 1.3 then coincide with point A, but the expression $\pm\sqrt{b^2-c^2}$ has E and F representing, respectively, $\pm\sqrt{9}$.

4c. A faithful representation of any number system would be defective if two different numbers were represented by the same point (part a) or if two different points represented the same number (part b).

6a. Substituting $z=x-\frac{a_2}{2}=x+2$ into $z^3-6z^2-3z+18=0$ reduces the equation to $x^3-15x-4=0$. Solving via Equation (1-3) gives $x=4$. Factoring then reveals $x^3-15x-4=(x-4)(x^2+4x+1)$. Solving $x^2+4x+1=0$ with the quadratic formula produces $x=-2\pm\sqrt{3}$. Thus, the three solutions are $z=6$, $\sqrt{3}$, and $-\sqrt{3}$.

8. Multiplying $\mathbf{i}$ with the standard unit vector would yield an angular displacement of zero radians, which would mean that $(\mathbf{i})(\mathbf{1})=\mathbf{1}$. The product $(-\mathbf{1})(-\mathbf{1})$ would have an angular displacement of π^2 radians, so would equal $\cos(\pi^2)+i\sin(\pi^2)\approx-0.9-0.43i$. Finally, under this scheme we would have $(-\mathbf{1})(\mathbf{1})=\mathbf{1}=(\mathbf{1})(\mathbf{1})$, but clearly $-\mathbf{1}\neq\mathbf{1}$.

Section 1.2. The Algebra of Complex Numbers: page 19

2a. $\overline{(1+i)2+1}(3+i)=(1-3i)(3+i)=6-8i$.

2c. $\operatorname{Re}\left[(i-1)^3\right]=\operatorname{Re}(2+2i)=2$.

2e. $\frac{1+2i}{3-4i}-\frac{4-3i}{2-i}=\frac{3+28i}{11-5i}=\frac{-107+323i}{146}=-\frac{107}{146}+\frac{323}{146}i$.

2g. $\operatorname{Re}\left[(x-iy)^2\right]=\operatorname{Re}\left(x^2-2xyi-y^2\right)=x^2-y^2$.

2i. $\operatorname{Re}\left[(x+iy)(x-iy)\right]=\operatorname{Re}\left(x^2+y^2\right)=x^2+y^2$.

4. Identity (1-10): Let $z=x+iy$. Then $\overline{z}=x-iy$, so $\overline{\overline{z}}=x-(-iy)=x+iy=z$. Similarly for the other identities.

6a. $\operatorname{Re}(z_1+z_2)=\operatorname{Re}[(x_1+iy_1)+(x_2+iy_2)]=\operatorname{Re}[(x_1+x_2)+i(y_1+y_2)]=x_1+x_2=\operatorname{Re}(z_1)+\operatorname{Re}(z_2)$.

6b. Of course, the identity is not true in general, and students need only produce one counterexample, such as $\operatorname{Re}[(2+i)(3+i)]=\operatorname{Re}(5+5i)=5$, but $\operatorname{Re}(2+i)\operatorname{Re}(3+i)=(2)(3)=6$. You might return to this question after polar form is discussed to show that the identity is true iff at least one of the numbers is real. Here's a proof: $\operatorname{Re}(z_1z_2)=\operatorname{Re}\left(r_1r_2e^{i(\theta_1+\theta_2)}\right)=r_1r_2\cos(\theta_1+\theta_2)$. $\operatorname{Re}(z_1)\operatorname{Re}(z_1)=r_1r_2\cos\theta_1\cos\theta_2$. The two expressions are equal provided $\cos(\theta_1+\theta_2)=\cos\theta_1\cos\theta_2$. As $\cos(\theta_1+\theta_2)=\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2$, equality occurs iff $\sin\theta_1=0$, or $\sin\theta_2=0$, i.e., iff at least one of the numbers is real.

8. For $n = 1$ we get $(z+w)^1 = w + z = \sum_{k=0}^{1} \binom{1}{k} z^k w^{1-k}$. Assume, for some $n \geq 1$, that $(z+w)^n = \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k}$. Using the standard combinatorial identity $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$, we deduce

$$\begin{aligned}
(z+w)^{n+1} &= (z+w)^n (z+w) = \left[\sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k}\right](z+w) \\
&= \sum_{k=0}^{n} \binom{n}{k} z^{k+1} w^{n-k} + \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k+1} \\
&= \sum_{k=0}^{n} \binom{n}{k} z^{k+1} w^{n-k} + \sum_{k=-1}^{n-1} \binom{n}{k+1} z^{k+1} w^{n-k} \\
&= \sum_{k=0}^{n-1} \binom{n}{k} z^{k+1} w^{n-k} + z^{n+1} + w^{n+1} + \sum_{k=0}^{n-1} \binom{n}{k} z^{k+1} w^{n-k} \\
&= \sum_{k=0}^{n-1} \left[\binom{n}{k} + \binom{n}{k+1}\right] z^{k+1} w^{n-k} + z^{n+1} + w^{n+1} \\
&= \sum_{k=0}^{n-1} \binom{n+1}{k+1} z^{k+1} w^{n-k} + z^{n+1} + w^{n+1} \\
&= \sum_{k=1}^{n} \binom{n+1}{k} z^k w^{n+1-k} + z^{n+1} + w^{n+1} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} z^k w^{n+1-k}.
\end{aligned}$$

10. If $(0,0)$ were to have a multiplicative inverse, say (x,y), then we would have $(0,0)(x,y) = (1,0)$. But according Definition 1.3, $(0,0)(x,y) = ((0)(x) - (0)(y), (0)(y) + (x)(0)) = (0,0)$ no matter what (x,y) is.

12. Following the hint, and using Definition 1.4, we get
$z^{-1} = \frac{(1,0)}{(x,y)} = \left(\frac{(1)(x)+(0)(y)}{x^2+y^2}, \frac{-(1)(y)+(x)(0)}{x^2+y^2}\right) = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$.
Thus, by Definition 1.3 we have

$$\begin{aligned}
zz^{-1} &= (x,y)\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right) \\
&= \left(x\left(\frac{x}{x^2+y^2}\right) - y\left(\frac{-y}{x^2+y^2}\right), x\left(\frac{-y}{x^2+y^2}\right) + \left(\frac{x}{x^2+y^2}\right)y\right) \\
&= \left(\frac{x^2+y^2}{x^2+y^2}, \frac{-xy+xy}{x^2+y^2}\right) = (1,0)\,.
\end{aligned}$$

14. Mimicking the argument for multiplication on page 16, we get
$\frac{x_1}{x_2} = \frac{(x_1,0)}{(x_2,0)} = \left(\frac{(x_1)(x_2)+(0)(0)}{x_2^2+0^2}, \frac{(-x_1)(0)+(x_2)(0)}{x_2^2+0^2}\right) = \left(\frac{x_1}{x_2}, 0\right)$.

Section 1.3. The Geometry of Complex Numbers: page 25

4b. $|z|^2 = |x+iy|^2 = \left(\sqrt{x^2+y^2}\right)^2 = x^2 + y^2 = (x+iy)(x-iy) = z\overline{z}$.

6a. This is the circle of radius 2 centered at $(-1+2i)$.

6c. This is the closed disk of radius 1 centered at $-2i$.

8. Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$. The midpoint of the line joining these two points is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$, which is the point $\frac{z_1+z_2}{2}$.

10. Let $z = x + iy$. Then $|z| = 0$ iff $\sqrt{x^2+y^2} = 0$ iff $x = 0$, and $y = 0$ iff $z = 0$.

12. Let $z = x + iy = (x, y)$. Clearly $iz = (-y, x)$, $-z = (-x, -y)$, and $-iz = (y, -x)$. The distance between each pair of points, taken in order, is in each case $\sqrt{(x+y)^2 + (x-y)^2}$. To show the lines connecting these points are perpendicular simply observe that the slopes of the lines connecting each pair of points, again taken in order, are negative reciprocals of each other. For example, the slope between z and iz is $\frac{y-x}{x+y}$, whereas the slope between iz and $-z$ is $\frac{x+y}{-y+x}$. You may wish to return to this exercise after Section 1.4, where it is learned that multiplication by i rotates a point by one right angle. Then a simple argument shows $|z| = |iz| = |-z| = |-iz|$, and so the equality of the sides follows because the sides of the square each are the hypotenuse of the right triangle having the vectors corresponding to the given points as their side.

14. The non-zero vectors $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are parallel iff there is a non-zero real number a such that $z_1 = az_2$ iff $x_1 = ax_2$ and $y_1 = ay_2$ iff $x_1y_2 = (ax_2)\,y_2$ and $x_2y_1 = x_2\,(ay_2) = (ax_2)\,y_2$ iff $x_1y_2 = x_2y_1$ iff $x_2y_1 - x_1y_2 = 0$. But $x_2y_1 - x_1y_2 = \operatorname{Im}(z_1\overline{z_2})$.

16. If $n = 0$, the expression is clearly true for all $z \neq 0$ (0^0, of course, is undefined). For positive integers, an easy induction argument works. When $n = 1$ the expression is clearly true. Assume, for some $k \geq 1$, that $\left|z^k\right| = |z|^k$. Then, (using Equation 1-25), $\left|z^{k+1}\right| = \left|z^k z\right| = \left|z^k\right||z| = |z|^{k+1}$. For negative integers, use the result for positive integers to derive $\left|z^{-n}\right| = \frac{1}{|z^n|} = \frac{1}{|z|^n} = |z|^{-n}$.

20. Let $z_1 = x_1 + iy_1$, and $z_2 = x_2 + iy_2$. Then $z_1\overline{z_2} + \overline{z_1}z_2 =$
$(x_1 + iy_1)(x_2 - iy_2) + (x_1 - iy_1)(x_2 + iy_2) =$
$[x_1x_2 + y_1y_2 + i(x_2y_1 - x_1y_2)] + [x_1x_2 + y_1y_2 + i(-x_2y_1 + x_1y_2)] =$
$2(x_1x_2 + y_1y_2)$, which is a real number.

22. Using standard techniques, we have $|z_1 - z_2|^2 = (z_1 - z_2)\overline{(z_1 - z_2)} = (z_1 - z_2)(\overline{z_1} - \overline{z_2}) = z_1\overline{z_1} - z_1\overline{z_2} - \overline{z_1}z_2 + z_2\overline{z_2} =$
$|z_1|^2 - z_1\overline{z_2} - \overline{z_1\overline{z_2}} + |z_2|^2 = |z_1|^2 - \left(z_1\overline{z_2} + \overline{z_1\overline{z_2}}\right) + |z_2|^2 =$
$|z_1|^2 - 2\operatorname{Re}(z_1\overline{z_2}) + |z_2|^2$.

24a. By definition a hyperbola is the locus of points z with the property that the difference of the distances between (z and z_1) and (z and z_2) is a constant This definition conforms exactly to the set theoretic description $\{z : |z - z_1| - |z - z_2| = K\}$, where z_1 and z_2 are the foci. Note: the distance between z_1 and z_2 must be less than K, because if $K \geq |z_1 - z_2|$, then $K \geq |z_1 - z_2| = |z_1 - z + z - z_2| \geq |z - z_1| - |z - z_2| = K$. Equality is possible only at points along the same line passing through z_1 and z_2, and inequality, of course, is impossible.

24b. With foci of ± 2 we get $\{z : |z+2| - |z-2| = K\} =$
$\left\{(x,y) : \sqrt{(x+2)^2 + y^2} - \sqrt{(x-2)^2 + y^2} = K\right\}$. Because $2 + 3i$ is on the hyperbola, we have $K = \sqrt{16+9} - \sqrt{9} = 2$. Squaring twice and combining terms gives $\left\{(x,y) : 3x^2 - y^2 = 3\right\}$.

24c. Letting $z_1 = -25$, and $z_2 = 25$, we see that $K = |7+24i+25| - |7+24i-25| = 40 - 30 = 10$. Then, with $z = (x, y)$, the equation of the hyperbola is $\sqrt{(x+25)^2 + y^2} - \sqrt{(x-25)^2 + y^2} = 10$. Squaring both sides, simplifying, squaring again, and simplifying again gives $-2400 + 96x^2 - 4y^2 = 0$. In standard form, $x^2 - \frac{y^2}{24} = 25$.

26. The reason is that $|z_1\overline{z_2}| = |z_1||\overline{z_2}|$ and that $|\overline{z_2}| = |z_2|$.

Section 1.4. The Geometry of Complex Numbers, Continued: page 34

2a. $\left(\sqrt{3}-i\right)\left(1+i\sqrt{3}\right) = 2e^{-i\frac{\pi}{6}}2e^{i\frac{\pi}{3}} = 4e^{i\frac{\pi}{6}} = 4\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right) = 2\sqrt{3}+2i.$

2c. $2i\left(\sqrt{3}+i\right)\left(1+i\sqrt{3}\right) = 2e^{i\frac{\pi}{2}}2e^{i\frac{\pi}{6}}2e^{i\frac{\pi}{3}} = 8e^{i\pi} = -8.$

4. For $z_1 \neq 0 \neq z_2$, let $\theta \in \arg z_1 + \arg z_2$. Then $\theta = \theta_1 + \theta_2$ for some $\theta_1 \in \arg z_1$ and some $\theta_2 \in \arg z_2$. Thus, $z_1 = |z_1|\, e^{i\theta_1}$, and $z_2 = |z_2|\, e^{i\theta_2}$. This gives $z_1 z_2 = |z_1|\,|z_2|\, e^{i(\theta_1+\theta_2)}$, so that $\theta = \theta_1 + \theta_2 \in \arg z_1 z_2$.

6. For $z_1 \neq 0 \neq z_2$, suppose $\arg z_1 = \arg z_2$. Then for any $\theta \in \arg z_1$ (or $\arg z_2$) we have $z_1 = |z_1|\, e^{i\theta}$, and $z_2 = |z_2|\, e^{i\theta} = \frac{|z_2|}{|z_1|}\,|z_1|\, e^{i\theta} = cz_1$, where $c = \frac{|z_2|}{|z_1|}$. Conversely, suppose $z_2 = cz_1$. Since c is a positive real constant, we have $|z_2| = |cz_1| = c\,|z_1|$. If $\theta \in \arg z_1$, then $z_1 = |z_1|\, e^{i\theta}$, so $z_2 = cz_1 = c\,|z_1|\, e^{i\theta} = |cz_1|\, e^{i\theta} = |z_2|\, e^{i\theta}$. This gives $\theta \in \arg z_2$, so that $\arg z_1 \subseteq \arg z_2$. A similar argument shows $\arg z_2 \subseteq \arg z_1$.

8. Theorem 1.3 gives $\operatorname{Arg} z_1 + \operatorname{Arg} z_2 \in \arg z_1 + \arg z_2 = \arg z_1 z_2$. From the given inequalities we conclude $-\pi < \operatorname{Arg} z_1 + \operatorname{Arg} z_2 \leq \pi$, which means $\operatorname{Arg} z_1 + \operatorname{Arg} z_2 = \operatorname{Arg} z_1 z_2$. The points satisfying the given inequalities are located in the right half plane, including the positive y-axis, but excluding the origin and the negative y-axis.

10. For $z_1 \neq 0 \neq z_2$, we know by Theorem 1.3 that $\arg \frac{z_1}{z_2} = \arg\left(z_1 \frac{1}{z_2}\right) = \arg z_1 + \arg \frac{1}{z_2}$, so all we need show is that $\arg \frac{1}{z} = -\arg z$. For $z \neq 0$, let $\theta \in \arg \frac{1}{z}$. Then $\frac{1}{z} = \left|\frac{1}{z}\right| e^{i\theta}$, so $z = |z|\, e^{-i\theta} = |z|\, e^{i(-\theta)}$. This shows $\theta \in -\arg z$, so that $\arg \frac{1}{z} \subseteq -\arg z$ Likewise, if $\theta \in -\arg z$, then $z = |z|\, e^{i(-\theta)}$, so that $\frac{1}{z} = \left|\frac{1}{z}\right| e^{i\theta}$, giving $\theta \in \arg \frac{1}{z}$.

12. For $z_1 \neq 0 \neq z_2$, we know by Theorem 1.3 that $\arg(z_1\overline{z_2}) = \arg z_1 + \arg \overline{z_2}$, so all we need show is that $\arg \overline{z} = -\arg z$. For $z \neq 0$, let $\theta \in \arg \overline{z}$. Then $\overline{z} = |\overline{z}|\, e^{i\theta} = |\overline{z}|\,(\cos\theta + i\sin\theta)$. This implies $z = \overline{\overline{z}} = |\overline{z}|\,(\cos\theta - i\sin\theta) = |\overline{z}|\cos(-\theta) + i\sin(-\theta) = |z|\, e^{i(-\theta)}$, so that $\theta \in -\arg z$. The proof that $-\arg z \subseteq \arg \overline{z}$ is similar. Alternatively, we can use exercise 6 and 10: Noting that $z_2\overline{z_2}$ is a positive constant, we have $\arg(z_1\overline{z_2}) = \arg\left(\frac{z_1}{z_2} z_2 \overline{z_2}\right) = \arg \frac{z_1}{z_2} = \arg z_1 - \arg z_2$.

14. In the figure, β is an argument of vector $z_2 - z_1$, and γ is an argument of vector $z_3 - z_1$. Thus, $\alpha = \beta - \gamma \in \arg(z_2 - z_1) - \arg(z_3 - z_1) = \arg \frac{z_2 - z_1}{z_3 - z_1}$.

Section 1.5. The Algebra of Complex Numbers, Revisited: page 41

2. $\left(\sqrt{3}+i\right)^2 = 2 + 2i\sqrt{3}$; $\left(2+2i\sqrt{3}\right)^2 = -8 + i8\sqrt{3}$.

4. If $z = 0$, then trivially $z^n + \overline{z}^n = 0$ is real. If $z = re^{i\theta} \neq 0$, then $z^n = r^n e^{in\theta}$, and $\overline{z}^n = r^n e^{-in\theta}$. Thus, $z^n + \overline{z}^n = r^n\left(e^{in\theta} + e^{-in\theta}\right) = r^n\left[(\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta)\right] = 2r^n\cos n\theta$, which is real.

6. Assuming $z = re^{i\theta} \neq 0$, the solution follows at once using Equation (1-45).

8. This is similar to Exercise 13 except that we observe z_i $(i = 1, \ldots, n-1)$ is a solution to $1 + z + z^2 + \cdots + z^{n-1} = 0$, as the last equation is equivalent to $\frac{1+z^n}{1-z} = 0$ for $z \neq 1$. Thus, z_i $(i = 1, \ldots, n-1)$ is a factor of $1 + z + z^2 + \cdots + z^{n-1}$. The result now follows from Theorem 1.4.

10a. $(\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta = \cos 2\theta + i\sin 2\theta$.

10b. The case for $n = 1$ is trivial. Suppose, for some $n \geq 1$, that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$. Then $(\cos\theta + i\sin\theta)^{n+1} = (\cos\theta + i\sin\theta)^n(\cos\theta + i\sin\theta) = (\cos n\theta + i\sin n\theta)(\cos\theta + i\sin\theta) = \cos n\theta\cos\theta - \sin n\theta\sin\theta + i(\sin n\theta\cos\theta + cosn\theta\sin\theta) = \cos(n+1)\theta + i\sin(n+1)\theta$.

Section 1.6. The Topology of Complex Numbers: page 50

2. We get $x = y^2 - 1$, which is a parabola oriented sideways with vertex at $(-1, 0)$ and opening to the right. For $-1 \le t \le 0$, gives the portion from $(-1, 0)$ to $(0, 1)$. For $1 \le t \le 2$, we get the portion from $(3, 2)$ to $(8, 3)$.

4. Let $z \in D_\varepsilon(z_0)$. With $z = (x, y)$, and $z_0 = (x_0, y_0)$, $z \in D_\varepsilon(z_0)$ translates to $\sqrt{(x - x_0)^2 + (y - y_0)^2} < x_0$. Squaring and rearranging terms gives $2xx_0 > x^2 + (y - y_0)^2 > 0$. Since $x_0 > 0$, this gives $x = \operatorname{Re} z > 0$.

6. Let z_1 and z_2 belong to $D_1(0)$, and let $z = z_1 + t(z_2 - z_1)$ be any point on the straight line segment joining z_1 to z_2, where $0 \le t \le 1$. Then $|z| = |z_1 + t(z_2 - z_1)| = |z_1(1 - t) + z_2 t| \le |z_1|(1 - t) + |z_2| t < (1 - t) + t = 1$, so that $z \in D_1(0)$. This shows $D_1(0)$ is connected. To show it is open, let $z_0 \in D_1(0)$, and let $\varepsilon = 1 - |z_0| > 0$. Suppose $z \in D\varepsilon(z_0)$. Then $|z| - |z_0| \le |z - z_0| < \varepsilon = 1 - |z_0|$. Thus, $|z| < 1$, so $z \in D_1(0)$. To show $\overline{D}_1(0)$ is not a domain, pick $z = 1 \in \overline{D}_1(0)$, and let $\varepsilon > 0$ be given. Then $1 + \frac{\varepsilon}{2} \in D_\varepsilon(1)$, but $1 + \frac{\varepsilon}{2}$ lies outside $\overline{D}_1(0)$.

8a. Let $z_0 \in K = \{z : |z| > 1\}$, and let $\varepsilon = |z_0| - 1 > 0$. Suppose $z \in D_\varepsilon(z_0)$. then $|z_0| - |z| \le |z - z_0| < \varepsilon = |z_0| - 1$. Thus, $|z| > 1$, so $z \in K$.

10. See the first part of the argument for exercise 6.

12. Clearly $C_\varepsilon(z_0)$ is contained in the boundary of $D_\varepsilon(z_0)$, as any neighborhood of any point in $C_\varepsilon(z_0)$ contains points that belong and do not belong to $D_\varepsilon(z_0)$. Also, if $z_0 \notin C_\varepsilon(z_0)$, then either $z_0 \in D_\varepsilon(z_0)$ or $z_0 \in (\mathbb{C} \smallsetminus D_\varepsilon(z_0))$. In either case, there is a neighborhood about z_0 that is contained in these sets. Showing this rigorously is a good exercise for interested students.

14. First, 0 is an accumulation point of the set $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ because (by the Archimedian property) for any $\varepsilon > 0$, there exists $n > 0$ such that $\frac{1}{n} < \varepsilon$, so $\frac{1}{n} \in D^*_\varepsilon(0)$. Also, for any $z_0 \ne 0$, there exists $\varepsilon > 0$ such that no point in $D^*_\varepsilon(z_0)$ belongs to S. Again, showing this last statement with rigor is a good exercise for interested students.

Chapter 2

Section 2.1. Functions and Linear Mappings: page 65

2a. $f(-1 + i) = f\left(\sqrt{2}e^{i\frac{3\pi}{4}}\right) = \left(\sqrt{2}e^{i\frac{3\pi}{4}}\right)^{21} - 5\left(\sqrt{2}e^{i\frac{3\pi}{4}}\right)^7 + 9\left(\sqrt{2}e^{i\frac{3\pi}{4}}\right)^4 = 1024\sqrt{2}e^{i\frac{63\pi}{4}} - 5\left(8\sqrt{2}e^{i\frac{21\pi}{4}}\right) + 9\left(4e^{i\frac{12\pi}{4}}\right) = 1024\sqrt{2}\left[\frac{\sqrt{2}}{2}(1 - i)\right] - 40\sqrt{2}\left[\frac{\sqrt{2}}{2}(-1 - i)\right] - 36 = 1024(1 - i) - 40(-1 - i) - 36 = 1028 - 984i$.

2b. Using the same procedure as in 2a, we get $f(1 + i\sqrt{3}) = -2,097,544 - 392i\sqrt{3}$.

4a. $f\left(re^{i\theta}\right) = r^5 e^{i5\theta} + r^5 e^{-i5\theta} = 2r^5 \cos 5\theta$.

4b. $f\left(re^{i\theta}\right) = r^5 \cos 5\theta + r^3 \cos 3\theta + i\left(r^5 \sin 5\theta - r^3 \sin 3\theta\right)$.

4c. The expressions are valid for all $z \ne 0$.

6a. $f(1) = 0$.

6c. $f(1 + i\sqrt{3}) = \ln 2 + \frac{i\pi}{3}$.

8a. Suppose $f(z_1) = f(z_2)$. Thus $g(f(z_1)) = g(f(z_2))$. But by Equations (2-3), $g(f(z_1)) = z_1$, and $g(f(z_2)) = z_2$. Thus, $z_1 = z_2$, so f is one-to-one.

8b. Let $b \in B$. Then $g(b) \in A$. By Equations (2-3), $f(g(b)) = b$, so f is an onto map.