

# Chapter 1

**Note:** Solutions not appearing in this complete solutions manual can be found in the student study guide.

## Exercises 1.1

$$\begin{aligned}
 2.a \quad & 2i^3 - 3i^2 + 5i \\
 & = 2i^2 \cdot i - 3i^2 + 5i = -2i - 3(-1) + 5i \\
 & = -2i + 3 + 5i = 3 + 3i
 \end{aligned}$$

$$\begin{aligned}
 2.b \quad & 3i^5 - i^4 + 7i^3 - 10i^2 - 9 \\
 & = 3i^4 \cdot i - i^4 + 7i^2 \cdot i - 10i^2 - 9 \\
 & = 3i - 1 - 7i + 10 - 9 \\
 & = -4i
 \end{aligned}$$

$$\begin{aligned}
 2.c \quad & \frac{5}{i} + \frac{2}{i^3} - \frac{20}{i^{18}} \\
 & = \frac{5i}{i^2} + \frac{2}{i^2 \cdot i} - \frac{20}{(i^2)^9} = -5i - \frac{2}{i} + 20 \\
 & = -5i - \frac{2i}{i^2} + 20 = -5i + 2i + 20 = 20 - 3i
 \end{aligned}$$

$$\begin{aligned}
 2.d \quad & 2i^6 + \left(\frac{2}{-i}\right)^3 + 5i^{-5} - 12i \\
 & = 2(i^2)^3 + \frac{8}{(-i)^3} + \frac{5}{i^5} - 12i = -2 - \frac{8}{i^2 \cdot i} + \frac{5}{i^4 \cdot i} - 12i \\
 & = -2 + \frac{8}{i} + \frac{5}{i} - 12i = -2 + \frac{8i}{i^2} + \frac{5i}{i^2} - 12i \\
 & = -2 - 8i - 5i - 12i = -2 - 25i
 \end{aligned}$$

$$3. \quad (5 - 9i) + (2 - 4i) = (5 + 2) + (-9 - 4)i = 7 - 13i$$

$$\begin{aligned}
 4. \quad & 3(4 - i) - 3(5 + 2i) \\
 & = 12 - 3i - 15 - 6i = -3 - 9i
 \end{aligned}$$

$$\begin{aligned}
 6. \quad & i(4 - i) + 4i(1 + 2i) \\
 & = 4i - i^2 + 4i + 8i^2 = ri + 1 + 4i - 8 = -7 + 8i
 \end{aligned}$$

$$7. \quad (2 - i)(4 + i) = 8 - 12i + 2i - 3i^2 = 8 - 10i + 3 = 11 - 10i$$

8. 
$$\begin{aligned} & \left(\frac{1}{2} - \frac{1}{4}i\right)\left(\frac{2}{3} + \frac{5}{3}i\right) \\ &= \frac{1}{3} + \frac{5}{6}i - \frac{1}{6}i - \frac{5}{12}i^2 = \frac{1}{3} + \frac{4}{6}i + \frac{5}{12} = \frac{4+5}{12} + \frac{2}{3}i \\ &= \frac{9}{12} + \frac{2}{3}i = \frac{3}{4} + \frac{2}{3}i \end{aligned}$$
10. 
$$\frac{i}{1+i} = \frac{i}{1+i} \cdot \frac{1-i}{1-i} = \frac{i-i^2}{(1)^2 - (i^2)} = \frac{i+1}{2} = \frac{1}{2} + \frac{i}{2}$$
11. 
$$\frac{2-4i}{3+5i} = \frac{2-4i}{3+5i} \cdot \frac{3-5i}{3-5i} = \frac{6-10i-12i+20i^2}{9-15i+15i-25i^2} = \frac{-14-22i}{34} = -\frac{7}{17} - \frac{11}{17}i$$
12. 
$$\frac{10-5i}{6+2i} = \frac{5(2-i)}{2(3+i)} = \frac{5(2-i)(3-i)}{2(3+i)(3-i)} = \frac{5(6-2i-3i+i^2)}{2(9-i^2)}$$
14. 
$$\begin{aligned} \frac{(1+i)(1-2i)}{(2+i)(4-3i)} &= \frac{1-2i+i-2i^2}{8-6i+4i-3i^2} = \frac{3-i}{11-2i} = \frac{(3-i)(11+2i)}{11^2 - (2i)^2} \\ &= \frac{33+6i-11i-2i^2}{121+4} = \frac{35-5i}{125} = \frac{35}{125} - \frac{5i}{125} = \frac{7}{25} - \frac{i}{25} \end{aligned}$$
15. 
$$\begin{aligned} \frac{(5-4i)-(3+7i)}{(4+2i)+(2-3i)} &= \frac{(5-3)+(-4-7)i}{(4+2)+(2-3)i} \\ &= \frac{2-11i}{6-i} \\ &= \frac{2-11i}{6-i} \cdot \frac{6+i}{6+i} \\ &= \frac{12+2i-66i-11i^2}{36+6i-6i-i^2} \\ &= \frac{23-64i}{37} \\ &= \frac{23}{37} - \frac{64}{37}i \end{aligned}$$
16. 
$$\begin{aligned} \frac{(4+5i)+2i^3}{(2+i)^2} &= \frac{4+5i+2i \cdot i^2}{4+i^2+4i} = \frac{4+3i}{4i+3} = \frac{(4+3i)(4i-3)}{(4i)^2 - 9} \\ &= \frac{16i-12+2i^2-9i}{-16-9} = \frac{7i-14}{-25} = \frac{14}{25} - \frac{7i}{25} \end{aligned}$$

18.  $(1+i)^2(1-i)^3$   
 $= (1+i^2+2i)(1-i)^2(1-i)$   
 $= (2i)(1-2i+i^2)(1-i) = (2i)(-2i)(1-i)$   
 $= -4i^2(1-i) = 4(1-i) = 4-4i$
19.  $(3+6i) + (4-i)(3+5i) + \frac{1}{2-i} = (3+6i) + (12+20i-3i-5i^2) + \frac{1}{2-i} \frac{2+i}{2+i}$   
 $= (3+6i) + (17+17i) + \frac{2+i}{4+2i-2i-i^2}$   
 $= (3+6i)\frac{5}{5} + (17+17i)\frac{5}{5} + \frac{2+i}{5}$   
 $= \frac{15+30i}{5} + \frac{85+85i}{5} + \frac{2+i}{5}$   
 $= \frac{102}{5} + \frac{116}{5}i$
20.  $(2+3i)\left(\frac{2-i}{1+2i}\right)^2$   
 $\frac{2-i}{1+2i} = \frac{2-i}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{2-4i-i+2i^2}{1^2-(2i)^2} = \frac{-5i}{5} = -i$   
 $(2+3i)\left(\frac{2-i}{1+2i}\right)^2 = (2+3i)(-i)^2 = -2-3i$
22.  $\left(1-\frac{1}{2}i\right)^3$   
 $= (1)^3 + \frac{3}{1!}(1)^2 \cdot \left(-\frac{1}{2}i\right) + \frac{3(2)}{2!}(1)^1 \left(-\frac{1}{2}i\right)^2 + \left(-\frac{1}{2}i\right)^3$   
 $= 1 - \frac{3}{2}i + 3\frac{1}{4}i^2 - \frac{1}{8}i^3 = 1 - \frac{3}{2}i + \frac{3}{4}(-1) + \frac{1}{8}i$   
 $= 1 - \frac{3}{4} - \frac{3}{2}i + \frac{1}{8}i = \frac{1}{4} - \frac{11}{8}i$
23.  $(-2+2i)^5 = (-2)^5 + \frac{5}{1}(-2)^4(2i)^1 + \frac{20}{2}(-2)^3(2i)^2 + \frac{60}{6}(-2)^2(2i)^3 + \frac{120}{24}(-2)^1(2i)^4 + (2i)^5$   
 $= -32 + (5)(16)(2i) + (10)(-8)(-4) + (10)(4)(-8i) + (5)(-2)(16) + 32i$   
 $= -32 + 160i + 320 - 320i - 160 + 32i$   
 $= 128 - 128i$

$$\begin{aligned}
24. \quad (1+i)^8 &= (1)^8 + \frac{8}{1!}(1)^7(i) + \frac{8(7)}{2!}(1)^6(i)^2 + \frac{8(7)(6)}{3!}(1)^5(i)^3 \\
&\quad + \frac{8(7)(6)(5)}{4!}(1)^4(i)^4 + \frac{8(7)(6)(5)(4)}{5!}(1)^3(i)^5 + \\
&\quad \frac{8(7)(6)(5)(4)(3)}{6!}(1)^2(i)^6 + \frac{8(7)(6)(5)(4)(3)(2)}{7!}(1)(i)^7 + i^8 \\
&= 1 + 8i - 28 + 56i^3 + 70i^4 + 56i^5 + 28i^6 + 8i^7 + i^8 \\
&= 1 + 8i - 28 - 56i + 70 + 56i - 28 - 8i + 1 \\
&= 16
\end{aligned}$$

$$\begin{aligned}
26. \quad z &= \frac{1}{(1+i)(1-2i)(1+3i)} \\
(1+i)(1-2i)(1+3i) &= (1-2i+i-2i^2)(1+3i) \\
&= (3-i)(1+3i) = 3+9i-i-3i^2 = 6+8i \\
\frac{1}{6+8i} &= \frac{1}{6+8i} \cdot \frac{6-8i}{6-8i} = \frac{6-8i}{36-64i^2} = \frac{6-8i}{100} = \frac{6}{100} - \frac{8}{100}i \\
&= \frac{3}{50} - \frac{4}{50}i = \frac{3}{50} - \frac{2}{25}i \\
\operatorname{Re}(z) &= \frac{3}{50}; \quad \operatorname{Im}(z) = \frac{-2}{25}
\end{aligned}$$

$$\begin{aligned}
27. \quad \operatorname{Re}\left(\frac{1}{z}\right) &= \operatorname{Re}\left(\frac{1}{x+iy}\right) \\
&= \operatorname{Re}\left(\frac{1}{x+iy} \cdot \frac{x-iy}{x-iy}\right) \\
&= \operatorname{Re}\left(\frac{x-iy}{x^2-ixy+ixy+i^2y^2}\right) \\
&= \operatorname{Re}\left(\frac{x-iy}{x^2+y^2}\right) \\
&= \operatorname{Re}\left(\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}\right) \\
&= \frac{x}{x^2+y^2}
\end{aligned}$$

$$\begin{aligned}
 28. \quad z &= x + iy \\
 \operatorname{Re}(z^2) & \\
 z^2 &= (x + iy)^2 = x^2 + 2xyi + i^2y^2 = x^2 - y^2 + 2xyi \\
 \operatorname{Re}(z^2) &= x^2 - y^2
 \end{aligned}$$

$$\begin{aligned}
 30. \quad \bar{z}^2 &= (x - iy)^2 = x^2 + i^2y^2 - 2xyi = x^2 - y^2 - 2xyi \\
 \bar{z}^2 + z^2 &= x^2 - y^2 - 2xyi + x^2 - y^2 + 2xyi \\
 &= 2x^2 - 2y^2 \\
 \operatorname{Im}(\bar{z}^2 + z^2) &= 0
 \end{aligned}$$

$$\begin{aligned}
 31. \quad \operatorname{Re}(iz) &= \operatorname{Re}(i(x + iy)) \\
 &= \operatorname{Re}(ix + i^2y) \\
 &= \operatorname{Re}(ix - y) \\
 &= -y \\
 &= -\operatorname{Im}(z)
 \end{aligned}$$

$$\begin{aligned}
 32. \quad z &= x + iy \\
 iz &= ix + i^2y = -y + ix \\
 \operatorname{Im}(iz) &= x = \operatorname{Re}(z)
 \end{aligned}$$

$$\begin{aligned}
 34. \quad \operatorname{Re}(z^2) &= x^2 - y^2 \\
 &= (\operatorname{Re}(z))^2 - (\operatorname{Im}(z))^2
 \end{aligned}$$

$$35. \quad \text{If } z_1 = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \text{ then}$$

$$\begin{aligned}
 z_1^2 + i &= \left( -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \right)^2 + i \\
 &= \frac{2}{4} - \frac{2}{4}i - \frac{2}{4}i + \frac{2}{4}i^2 + i \\
 &= \frac{1}{2} - i - \frac{1}{2} + i \\
 &= 0.
 \end{aligned}$$

An additional solution is  $z_2 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ .

36.  $z^4 = -4$

$$z_1 = 1 + i, z_2 = -1 + i$$

$$z_3 = \frac{-4}{1+i} = \frac{-4}{1+i} \cdot \frac{1-i}{1-i} = \frac{-4+4i}{2} = -2+2i = 2(-1+i)$$

$$z_4 = \frac{-4}{-1+i} = \frac{-4}{-1+i} \cdot \frac{-1-i}{-1-i} = \frac{4+4i}{(1)^2 - i^2} = \frac{4+4i}{2} = 2(1+i)$$

38.  $z - 2\bar{z} + 7 - 6i = 0$

$$z - 2\bar{z} = 6i - 7$$

$$x + iy - 2(x - iy) = 6i - 7$$

$$x + iy - 2x + 2iy = 6i - 7$$

$$-x + 3iy = 6i - 7$$

$$-x = -7 \Rightarrow x = 7$$

$$3y = 6 \Rightarrow y = 2$$

39. Let  $z = a + ib$ , then  $z^2 = (a + ib)^2 = a^2 + iab + iab + i^2b^2 = a^2 - b^2 + 2abi$ . By Definition 1.1.2,  $z^2 = i$  if and only if  $\text{Re}(z^2) = \text{Re}(i)$  and  $\text{Im}(z^2) = \text{Im}(i)$ . This gives the following two equations:

$$a^2 - b^2 = 0$$

$$2ab = 1.$$

From the second equation,  $b = \frac{1}{2a}$ . Substituting this into the first equation we get

$$a^2 - \frac{1}{4a^2} = 0 \quad \text{or} \quad a^4 = \frac{1}{4}.$$

Solving for  $a$  we find that  $a = \pm \frac{\sqrt{2}}{2}$ . Now since  $b = \frac{1}{2a}$ , we have two solutions,  $a =$

$$\frac{\sqrt{2}}{2}, b = \frac{\sqrt{2}}{2} \quad \text{and} \quad a = -\frac{\sqrt{2}}{2}, b = -\frac{\sqrt{2}}{2}. \quad \text{Therefore, } z = \pm \left( \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \right).$$

$$\begin{aligned}
40. \quad \bar{z}^2 &= 4z \\
(x - iy)^2 &= 4(x + iy) \\
x^2 + i^2 y^2 - 2xyi &= 4x + 4iy \\
x^2 - y^2 &= 4x \\
-2xy &= 4y \Rightarrow -2x = 4 \Rightarrow x = -2 \\
x^2 - y^2 = 4x &\Rightarrow 4 - y^2 = -8 \Rightarrow 12 = y^2 \Rightarrow y = \pm 2\sqrt{3}
\end{aligned}$$

$$42. \quad \frac{z}{1 + \bar{z}} = 3 + 4i$$

$$\frac{z}{1 + \bar{z}} = \frac{z(1 - \bar{z})}{1^2 - \bar{z}^2} = \frac{(x + iy)(1 - x + iy)}{1 - (x - iy)^2}$$

$$= \frac{x - x^2 + xyi + iy - xyi + i^2 y^2}{1 - (x^2 + i^2 y^2 - 2xyi)} = \frac{x - x^2 - y^2 + iy}{1 - x^2 + y^2 + 2xyi}$$

$$\frac{x - x^2 - y^2 + iy}{1 - x^2 + y^2 + 2xyi} = 3 + 4i$$

$$x - x^2 - y^2 + iy = (3 + 4i)(1 - x^2 + y^2 + 2xyi)$$

$$x - x^2 - y^2 + iy = 3 - 3x^2 + 3y^2 + 6xyi + 4i - 4ix^2 + 4iy^2 + 8xyi^2$$

$$x - x^2 - y^2 + iy = 3 - 3x^2 + 3y^2 + 6xyi + 4i - 4x^2i + 4y^2i - 8xy$$

$$x - x^2 - y^2 = 3 - 3x^2 + 3y^2 - 8xy$$

$$y = 6xy + 4 - 4x^2 + 4y^2$$

$$\begin{cases} 2x^2 - 4y^2 + x + 8xy - 3 = 0 \\ -4x^2 + 4y^2 - y + 6xy + 4 = 0 \end{cases}$$

43. To solve we multiply the first equation by  $-i$  then sum the two equations.

$$\begin{array}{rcl}
-i^2 z_1 - i^2 z_2 & = & -i(2 + 1 - i) \quad z_1 - z_2 = 10 - 2i \\
-z_1 + (1 - i)z_2 & = & 3 - 5i \quad \Rightarrow \quad \underline{-z_1 + (1 - i)z_2 = 3 - 5i} \\
& & -iz_2 = 13 - 7i
\end{array}$$

Thus,  $-iz_2 = 13 - 7i$  or  $z_2 = 7 + 13i$ . Now substitute this value of  $z_2$  into the first equation and solve for  $z_1$ .

$$\begin{aligned}
iz_1 - i(7 + 13i) &= 2 + 10i \\
iz_1 + 13 - 7i &= 2 + 10i \\
iz_1 &= -11 + 17i \\
z_1 &= 17 + 11i
\end{aligned}$$

$$44. \quad iz_1 + (1+i)z_2 = 1 + 2i$$

$$(2-i)z_1 + 2iz_2 = 4i$$

$$= \frac{-2i}{1+i}(iz_1) - 2iz_2 = (1+2i) \cdot \left( \frac{-2i}{1+i} \right) \left. \vphantom{\frac{-2i}{1+i}} \right\} \oplus$$

$$(2-i)z_1 + 2iz_2 = 4i$$

$$\left( \frac{2}{1+i} + 2-i \right) z_1 = \frac{-2i(1+2i)}{1+i} + 4i$$

$$(1-i+2-i)z_1 = -3i+1+4i$$

$$(3-2i)z_1 = 1+i \Rightarrow z_1 = \frac{1+i}{3-2i} \Rightarrow z_1 = \frac{1}{13} + \frac{5}{13}i$$

$$(2-i)z_1 + 2iz_2 = 4i$$

$$(2-i) \left( \frac{1}{13} + \frac{5}{13}i \right) + 2iz_2 = 4i$$

$$\frac{2}{13} + \frac{10}{13}i - \frac{1}{13} + \frac{5}{13} + 2iz_2 = 4i$$

$$\frac{7}{13} + \frac{9}{13}i + 2iz_2 = 4i$$

$$2iz_2 = 4i - \frac{7}{13} - \frac{9}{13}i$$

$$2iz_2 = -\frac{7}{13} + \frac{43}{13}i$$

$$z_2 = \frac{-\frac{7}{13} + \frac{43}{13}i}{2i}$$

$$= \frac{-\frac{43}{13} - \frac{7}{13}i}{-2} = \frac{43}{26} + \frac{7}{26}i$$

$$46. \quad (1+i)^8 = 16$$

$$(1+i)^{5404} = 10808$$

$$47. \quad (a) \text{ Since } i^4 = 1, \text{ it follows that } i^{4k} = (i^4)^k = 1^k = 1. \text{ So, } n = 4k.$$

$$48. \quad 5+6i = (1+u)(a+ib)$$

$$= a+ib+ai-b$$

$$5 + 6i = a - b + (a + b)i$$

$$\left. \begin{array}{l} a - b = 5 \\ a + b = 6 \end{array} \right\} \oplus \quad 2a = 11 \quad a = \frac{11}{2}$$

$$a + b = 6 \quad b = 6 - a = 6 - \frac{11}{2} = \frac{1}{2}$$

$$\frac{1}{3 - 4i} = a + ib$$

$$(a + ib)(3 - 4i) = 1 \Rightarrow 3a - 4ai + 3bi + 4b = 1$$

$$\Rightarrow \begin{cases} 3a + 4b = 1 \\ 3b - 4a = 0 \end{cases} \quad 3b = 4a \quad b = \frac{4}{3}a$$

$$3a + 4b = 1$$

$$3a + 4\left(\frac{4}{3}a\right) = 1 \quad 3a + \frac{16}{3}a = 1$$

$$\frac{25}{3}a = 1 \quad a = \frac{3}{25}$$

$$b = \frac{4}{3}a = \frac{4}{3} \cdot \frac{3}{25} = \frac{4}{25}$$

**50.** Suppose  $z_1 z_2 = 0$  and  $z_1 \neq 0$ . Then divide both sides of  $z_1 z_2 = 0$  by  $z_1$  to get  $z_2 = 0$ .

Thus, either  $z_1 = 0$  or  $z_2 = 0$ .

**51.** Suppose  $z_1 z_2 = r$  where  $r$  is a real number. Then  $z_1 = r/z_2$ . Now multiply the numerator and denominator of  $r/z_2$  by  $\bar{z}_2$  and explain why the result is of the form  $k\bar{z}_2$  where  $k$  is a real constant.

**52.**  $z_1 \bar{z}_1 + \bar{z}_1 z_2 = 2\operatorname{Re}(z_1 \bar{z}_2)$

$$z_1 = x_1 + iy_1 \quad \bar{z}_1 = x_1 - iy_1$$

$$z_2 = x_2 + iy_2 \quad \bar{z}_2 = x_2 - iy_2$$

$$z_1 \bar{z}_2 = (x_1 + iy_1)(x_2 - iy_2) = x_1 x_2 - x_1 y_2 i + x_2 y_1 i + y_1 y_2$$

$$\bar{z}_1 z_2 = (x_1 - iy_1)(x_2 + iy_2) = x_1 x_2 + x_1 y_2 i - x_2 y_1 i + y_1 y_2$$

$$z_1 \bar{z}_2 + \bar{z}_1 z_2 = 2x_1 x_2 + 2y_1 y_2$$

$$\operatorname{Re}(z_1 \bar{z}_2) = x_1 x_2 + y_1 y_2$$

$$2\operatorname{Re}(z_1 \bar{z}_2) = 2x_1 x_2 + 2y_1 y_2$$

**55.** Assume that such a subset  $P$  exists and that  $i$  is in  $P$ . By the second property (applied two times)  $i^3$  is in  $P$ . Explain why this leads to a contradiction.

## Exercises 1.2

$$\begin{aligned}
 2. \quad z_1 &= 1 - i \\
 z_2 &= 1 + i \\
 z_1 + z_2 &= 1 - i + 1 + i \\
 &= 2 \\
 z_1 - z_2 &= 1 - i - 1 - i \\
 &= -2i
 \end{aligned}$$

$$\begin{aligned}
 4. \quad z_1 &= 4 - 3i \\
 z_2 &= -2 + 3i \\
 2z_1 + 4z_2 &= 2(4 - 3i) + 4(-2 + 3i) \\
 &= 8 - 6i - 8 + 12i \\
 &= 6i \\
 z_1 - z_2 &= 4 - 3i + 2 - 3i \\
 &= 6 - 6i
 \end{aligned}$$

$$\begin{aligned}
 6. \quad z_1 &= -2 - 8i \\
 z_2 &= 3i \\
 z_3 &= -6 - 5i \\
 z_2 z_3 &= z_2 - z_3 \\
 z_1 z_2 &= z_1 - z_2 \\
 z_1 z_3 &= z_1 - z_3
 \end{aligned}$$

7. Two adjacent sides of the triangle are

$$\begin{aligned}
 z_1 - z_3 &= (-2 - 8i) - (-6 - 5i) = 4 - 3i, \text{ and} \\
 z_2 - z_3 &= (3i) - (-6 - 5i) = 6 + 8i.
 \end{aligned}$$

These sides are represented by the two vectors  $4i - 3j$  and  $6i + 8j$ . the angle  $\theta$  between these vectors can be determined using the dot product (see Review Topic: Vectors):

$$(4\mathbf{i} - 3\mathbf{j}) \cdot (6\mathbf{i} + 8\mathbf{j}) = 24 - 24 = 0.$$

This implies that  $\cos \theta = 0$ , and so  $\theta = \pi/2$ . Therefore,  $z_1$ ,  $z_2$ , and  $z_3$  are the vertices of a right triangle.

$$\begin{aligned}
 8. \quad z_1 &= 1 + 5i \\
 z_2 &= -4 - i \\
 z_3 &= 3 + i \\
 z_3 - z_2 &= 3 + i + 4 + i \\
 &= 7 + 2i \\
 d &= \sqrt{(7-1)^2 + (2-5)^2} \\
 &= \sqrt{6^2 + (-3)^2} \\
 &= \sqrt{36 + 9} = \sqrt{45} \\
 d &= 3\sqrt{5}
 \end{aligned}$$

$$\begin{aligned}
 10. \quad i(2-i) - 4\left(1 + \frac{1}{4}i\right) \quad |z| &= \sqrt{x^2 + y^2} \\
 2i - i^2 - 4 - i & \\
 = 2i + 1 - 4 - i & \\
 = i - 3 & \\
 |z| = \sqrt{(-3)^2 + (1)^2} &= \sqrt{9 + 1} = \sqrt{10}
 \end{aligned}$$

$$\begin{aligned}
 11. \quad \left| \frac{2i}{3-4i} \right| &= \frac{|2i|}{|3-4i|} \\
 &= \frac{\sqrt{(0)^2 + (2)^2}}{\sqrt{(3)^2 + (-4)^2}} \\
 &= \frac{\sqrt{4}}{\sqrt{25}} \\
 &= \frac{2}{5}
 \end{aligned}$$

$$\begin{aligned}
 12. \quad \frac{1-2i}{1+i} + \frac{2-i}{1-i} & \\
 = \frac{(1-2i)(1-i)}{(1)^2 - (i)^2} + \frac{(2-i)(1+i)}{(1)^2 - (i)^2} &= \frac{1-i-2i+2i^2}{1+1} + \frac{2+2i-i-i^2}{1+1} \\
 = \frac{-1-3i}{2} + \frac{3+i}{2} &= \frac{-1-3i+3+i}{2} = \frac{-2i+2}{2} = -i+1 \\
 |z| = \sqrt{(-1)^2 + (1)^2} &= \sqrt{1+1} = \sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 14. \quad & |z + \bar{5}| \\
 & z + \bar{5} = x + iy + 5(x - iy) \\
 & = x + iy + 5x - 5yi \\
 & = 6x - 4yi
 \end{aligned}$$

$$\begin{aligned}
 |z + 5\bar{z}| &= \sqrt{(6x)^2 + (-4y)^2} = \sqrt{36x^2 + 16y^2} \\
 &= \sqrt{4(9x^2 + 4y^2)} = 2\sqrt{9x^2 + 4y^2}
 \end{aligned}$$

15. In order to determine which complex number is closest to the origin we compute the distance from  $z_1$  to 0 and the distance from  $z_2$  to 0.

$$\begin{aligned}
 |z_1 - 0| &= |10 + 8i| = \sqrt{(10)^2 + (8)^2} = \sqrt{164} \\
 |z_2 - 0| &= |11 - 6i| = \sqrt{(11)^2 + (-6)^2} = \sqrt{157}
 \end{aligned}$$

Therefore,  $z_2$  is closest to the origin. On the other hand, the distance from  $z_1$  to  $1 + i$  and the distance from  $z_2$  to  $1 + i$  is:

$$\begin{aligned}
 |z_1 - (1 + i)| &= |9 + 7i| = \sqrt{(9)^2 + (7)^2} = \sqrt{130} \\
 |z_2 - (1 + i)| &= |10 - 7i| = \sqrt{(10)^2 + (-7)^2} = \sqrt{149}
 \end{aligned}$$

Therefore,  $z_1$  is closest to  $1 + i$ .

$$16. \quad \frac{1}{2} - \frac{1}{4}i \quad \frac{2}{3} + \frac{1}{6}i$$

$$d_1 = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{4}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{16}} = \sqrt{\frac{4+1}{16}} = \sqrt{\frac{5}{16}} = \sqrt{\frac{5}{4}}$$

$$d_2 = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{6}\right)^2} = \sqrt{\frac{4}{9} + \frac{1}{36}} = \sqrt{\frac{16+1}{36}} = \sqrt{\frac{17}{36}} = \sqrt{\frac{17}{6}}$$

$\frac{1}{2} - \frac{1}{4}i$  is closest to origin.

For  $1 + i$

$$d_1 = \sqrt{\left(1 - \frac{1}{2}\right)^2 + \left(1 + \frac{1}{4}\right)^2} = \sqrt{\frac{1}{4} + \frac{25}{16}} = \sqrt{\frac{4+25}{16}} = \sqrt{\frac{29}{16}} = \sqrt{\frac{29}{4}}$$

$$d_2 = \sqrt{\left(1 - \frac{2}{3}\right)^2 + \left(1 - \frac{1}{6}\right)^2} = \sqrt{\frac{1}{9} + \frac{25}{36}} = \sqrt{\frac{29}{36}} = \sqrt{\frac{29}{6}}$$

18.  $[\text{Im}(i\bar{z})]^2 = 2$

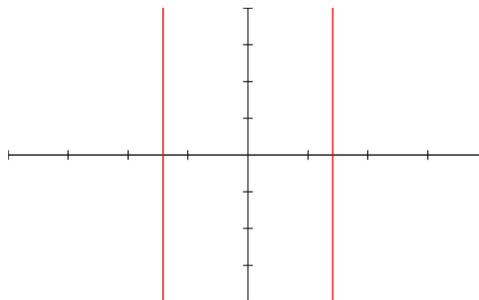
$$i\bar{z} = i(x - iy) = y + ix$$

$$\text{Im}(i\bar{z}) = x$$

$$[\text{Im}(i\bar{z})]^2 = x^2$$

$$\Rightarrow x^2 = 2$$

$$\Rightarrow x = \pm\sqrt{2}$$



19. If  $|z - i| = |z - 1|$ , then  $|z - i|^2 = |z - 1|^2$ . Let  $z = x + iy$ , then:

$$|(x + iy) - i|^2 = |(x + iy) - 1|^2$$

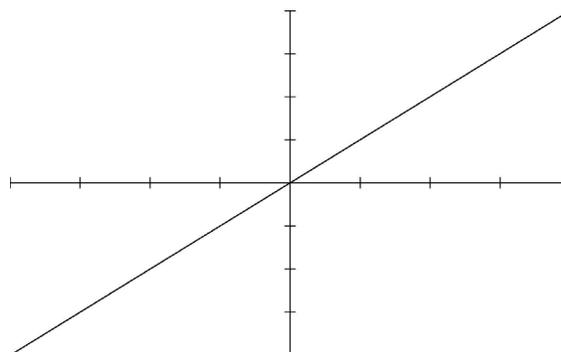
$$|x + i(y - 1)|^2 = |(x - 1) + iy|^2$$

$$x^2 + (y - 1)^2 = (x - 1)^2 + y^2$$

$$\cancel{x^2} + \cancel{y^2} - 2y + \cancel{1} = \cancel{x^2} - 2x + \cancel{1} + \cancel{y^2}$$

$$-2y = -2x$$

$$y = x.$$

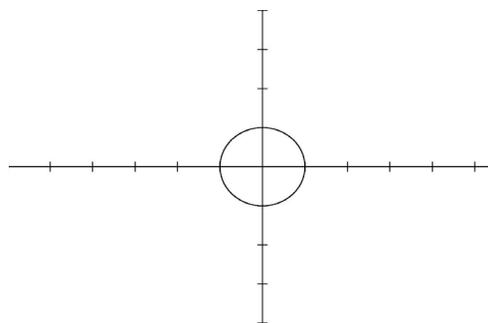


The set of points satisfying the equation  $|z - i| = |z - 1|$  lie on the line  $y = x$  in the plane.

20.  $\bar{z} = z^{-1}$

$$\bar{z} = \frac{1}{z} \Rightarrow z\bar{z} = 1 \Rightarrow x^2 + y^2 = 1$$

circle centered at origin with radius 1



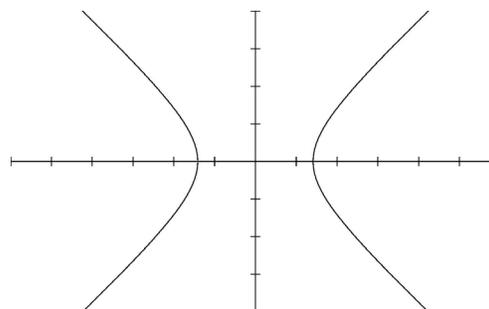
22.  $\text{Re}(z^2) = |\sqrt{3} - i|$

$$z^2 = (x + iy)^2 = x^2 - y^2 + 2xyi$$

$$\text{Re}(z^2) = x^2 - y^2$$

$$|\sqrt{3} - i| = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{3 + 1} = \sqrt{4} = 2$$

$$x^2 - y^2 = 2 \quad \text{Hyperbola.}$$



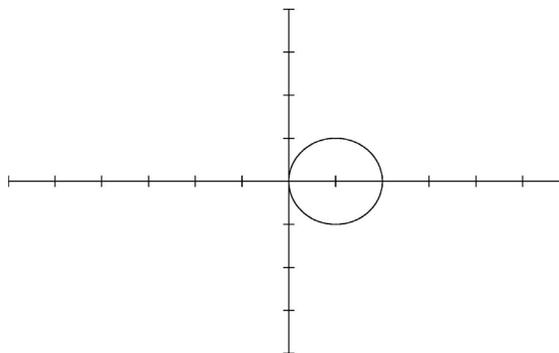
23. If  $|z-1|=1$ , then  $|z-1|^2=1$ .

Let  $z = x + iy$ , then:

$$|(x + iy) - 1|^2 = 1$$

$$|(x - 1) + iy|^2 = 1$$

$$(x - 1)^2 + y^2 = 1.$$



The set of points satisfying the equation  $|z-1|=1$  lie on the circle  $(x-1)^2 + y^2 = 1$  with center  $(1, 0)$  and radius 1.

24.  $|z-i|=2|z-1|$

$$z - i = x + iy - i = x + (y - 1)i$$

$$|z - i| = \sqrt{x^2 + (y - 1)^2}$$

$$z - 1 = x + iy - 1 = (x - 1) + yi$$

$$|z - 1| = \sqrt{(x - 1)^2 + y^2} \Rightarrow 2|z - 1| = 2\sqrt{(x - 1)^2 + y^2}$$

$$\sqrt{x^2 + (y - 1)^2} = 2\sqrt{(x - 1)^2 + y^2}$$

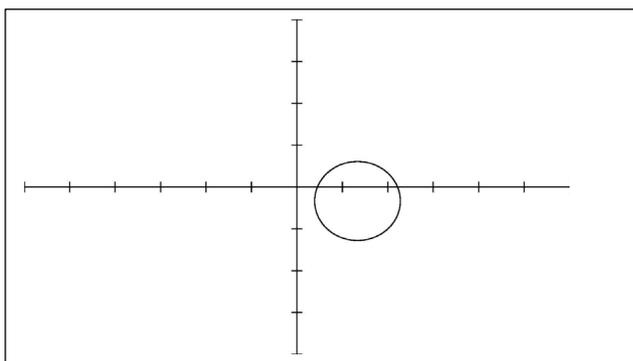
$$\Rightarrow x^2 + (y - 1)^2 = 4((x - 1)^2 + y^2)$$

$$\Rightarrow x^2 + y^2 - 2y + 1 = 4(x^2 - 2x + 1 + y^2)$$

$$\Rightarrow x^2 + y^2 - 2y + 1 = 4x^2 - 8x + 4 + 4y^2$$

$$\Rightarrow 3y^2 + 3x^2 - 8x + 2y + 3 = 0$$

because the coefficients of  $x^2$  and  $y^2$  are equal  $\Rightarrow$  circle.



26.  $|z| = \operatorname{Re}(z)$

$$\sqrt{x^2 + y^2} = x$$

$$x^2 + y^2 = x^2 \Rightarrow y^2 = 0 \Rightarrow y = 0 \Rightarrow x\text{-axis}$$

27. We use the triangle inequalities:

$$\left| |z| - |6 + 8i| \right| \leq |z + 6 + 8i| \leq |z| + |6 + 8i|.$$

Making the substitutions  $|z| = 2$  and  $|6 + 8i| = \sqrt{6^2 + 8^2} = 10$  we obtain

$$|2 - 10| \leq |z + 6 + 8i| \leq 2 + 10.$$

Thus,  $8 \leq |z + 6 + 8i| \leq 12$ .

28.  $|z| = 1$

$$\left| |z^2 - 3| \right| \leq 4$$

$$|z^2 - 3| \leq |z^2| + |-3|$$

$$|z^2 - 3| \leq 1 + 3$$

$$|z^2 - 3| \leq 4$$

30.  $z^4 - 5z^2 + 6 \quad |z| = 2$

$$z^4 - 5z^2 + 6 = (z^2 - 3)(z^2 - 2)$$

$$\left| \frac{1}{z^4 - 5z^2 + 6} \right| \leq M$$

$$|z^4 - 5z^2 + 6| = |(z^2 - 3)(z^2 - 2)| = |z^2 - 3| |z^2 - 2|$$

$$\leq (|z^2| + |-3|)(|z^2| + |-2|)$$

$$\leq (4 + 3)(4 + 2) \leq (7)(6) \leq 42$$

$$\Rightarrow \left| \frac{1}{z^4 - z^2 + 6} \right| \leq \frac{1}{42}$$

31. Let  $z = x + iy$ . From Definition 1.1.2,  $|z| - z = 2 + i$  if and only if  $\operatorname{Re}(|z| - z) = \operatorname{Re}(2 + i) = 2$  and  $\operatorname{Im}(|z| + z) = \operatorname{Re}(2 + i) = 1$ . This gives the following two equations:

$$\operatorname{Re}\left(\sqrt{x^2 + y^2} - (x + iy)\right) = \sqrt{x^2 + y^2} - x = 2$$

$$\operatorname{Im}\left(\sqrt{x^2 + y^2} - (x + iy)\right) = -y = 1.$$

Since  $y = -1$  from the second equation, the first equation becomes

$$\begin{aligned}\sqrt{x^2+1}-x &= 2 \\ \sqrt{x^2+1} &= 2+x \\ x^2+1 &= (2+x)^2 \\ \cancel{x^2}+1 &= 4+4x+\cancel{x^2} \\ -4x &= 3 \\ x &= -\frac{3}{4}\end{aligned}$$

The complex number  $z = -\frac{3}{4} - i$  satisfies the given equation.

**32.**  $|z|^2 + 1 + 12i = 6z$

$$x^2 + y^2 + 1 + 12i = 6(x + iy)$$

$$x^2 + y^2 + 1 + 12i = 6x + 6yi$$

$$\begin{cases} x^2 + y^2 + 1 = 6x \\ 12 = 6y \Rightarrow y = \frac{12}{6} = 2 \end{cases}$$

$$x^2 + y^2 + 1 = 6x$$

$$\Rightarrow x^2 + 4 + 1 = 6x \Rightarrow x^2 - 6x + 5 = 0$$

$$\Rightarrow (x-1)(x-5) = 0$$

$$\Rightarrow x = 1 \text{ or } x = 5$$

$$\begin{cases} z = 1 + 2i \\ \text{or} \\ z = 5 + 2i \end{cases}$$

**34.a**  $z = a + ib$

$$-z = -a - ib$$

symmetric with respect to origin

**34.b**  $z = a + ib$

$$z^{-1} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$$

symmetric with respect to  $x$ -axis

**35.** In each case consider the angle between  $z$  and  $iz$ .

36. Suppose  $|z| = 0$  and  $z = x + iy$ .

$$\sqrt{x^2 + y^2} = 0$$

$$x^2 + y^2 = 0$$

$$x = 0 \quad \text{and} \quad y = 0$$

Thus  $z = 0 + i0 = 0$ .

38.  $z = x + iy$       radius = 5  
center (3, -6)

$$|z^2| = x^2 + y^2$$

$$(x - 3)^2 + (y + 6)^2 = 25$$

$$x^2 - 6x + 9 + y^2 + 12y + 36 = 25$$

$$x^2 + y^2 - 6x + 12y + 45 = 25$$

$$x^2 + y^2 - 6x + 12y = -20$$

$$|z|^2 - 6x + 12y = 20i^2$$

39. Consider  $|z|$ .

42.  $z_1 = x_1 + iy_1$

$$z_2 = x_2 + iy_2$$

equation of line  $y - y_1 = m(x - x_1)$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

43. Rewrite the equation as  $(z_3 - z_2) = -k(z_1 - z_2)$ . Now both  $z_3 - z_2$  and  $z_1 - z_2$  can be viewed as vectors with the same initial point  $z_2$ . Since  $k$  is a real number,  $-k(z_1 - z_2)$  is just a scalar multiple of  $z_1 - z_2$ . Use the equality above to make a statement regarding the directions of these two vectors.

46.  $z = x + iy$

$$|z| = \sqrt{x^2 + y^2}$$

$$\operatorname{Re}(z) = x \Rightarrow |\operatorname{Re}(z)| = \sqrt{x^2} = x$$

$$x \leq \sqrt{x^2 + y^2} \quad \text{it could be equal if } y = 0$$

$$\operatorname{Im}(z) = y \Rightarrow |\operatorname{Im}(z)| = \sqrt{y^2} = y$$

$$y \leq \sqrt{x^2 + y^2} \quad \text{it could be equal if } x = 0$$

47. For both parts, set  $z = x + iy$  and use the definition of modulus.

48.  $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

$$z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2 = (x_1 + x_2) + i(y_1 + y_2)$$

$$|z_1 + z_2|^2 = (x_1 + x_2)^2 + (y_1 + y_2)^2$$

$$z_1 - z_2 = x_1 + iy_1 - x_2 - iy_2 = (x_1 - x_2) + i(y_1 - y_2)$$

$$|z_1 - z_2|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$z_1 = x_1 + iy_1 \Rightarrow |z_1|^2 = x_1^2 + y_1^2$$

$$z_2 = x_2 + iy_2 \Rightarrow |z_2|^2 = x_2^2 + y_2^2$$

$$2(|z_1|^2 + |z_2|^2) = 2(x_1^2 + y_1^2 + x_2^2 + y_2^2)$$

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2x_1^2 + 2y_1^2 + 2x_2^2 + 2y_2^2$$

## Exercises 1.3

2.  $-10$

$$x = -10 \quad y = 0$$

$$r = |z| = \sqrt{(-10)^2 + 0^2} = \sqrt{100} = 10$$

$$\frac{y}{x} = \frac{0}{-10} = 0 \quad \theta = \tan^{-1}(0) = \pi$$

$$z = r(\cos \theta + i \sin \theta) = 10(\cos \pi + i \sin \pi)$$

3. Since  $-3i = 0 + (-3)i$ , we identify  $x = 0$  and  $y = -3$ . Then  $\sqrt{0^2 + (-3)^2} = 3$ . Since  $x = 0$  and  $y = -3 < 0$ ,  $\text{Arg}(-3i) = -\frac{\pi}{2}$ . A different argument for  $-3i$  is given by  $\theta = \text{Arg}(-3i) + 2\pi = \frac{3\pi}{2}$ . Therefore,

$$-3i = 3 \left( \cos \left( -\frac{\pi}{2} \right) + i \sin \left( -\frac{\pi}{2} \right) \right), \quad \text{using } \theta = \text{Arg}(-3i)$$

$$-3i = 3 \left( \cos \left( \frac{3\pi}{2} \right) + i \sin \left( \frac{3\pi}{2} \right) \right), \quad \text{using } \theta \neq \text{Arg}(-3i).$$

4.  $6i$

$$x = 0 \quad y = 6$$

$$r = |z| = \sqrt{0^2 + (6)^2} = 6$$

$$\frac{y}{x} = \frac{6}{0} = \infty \quad \theta = \tan^{-1}(\infty) = \frac{\pi}{2}$$

$$z = r(\cos \theta + i \sin \theta) = 6 \left( \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)$$

6.  $5 - 5i$

$$x = 5 \quad y = -5$$

$$r = |z| = \sqrt{(5)^2 + (-5)^2} = \sqrt{50} = 5\sqrt{2}$$

$$\frac{y}{x} = \frac{-5}{5} = -1 \quad \theta = \tan^{-1}(-1) = \frac{-\pi}{4}$$

$$z = r(\cos \theta + i \sin \theta) = 5\sqrt{2} \left( \cos \left( \frac{-\pi}{4} \right) + i \sin \left( \frac{-\pi}{4} \right) \right)$$

7. For  $-\sqrt{3} + i$ , we identify  $x = -\sqrt{3}$  and  $y = 1$ . Then  $r = \sqrt{(\sqrt{3})^2 + 1^2} = 2$ . Since  $z = -\sqrt{3} + i$  is in the second quadrant,

$$\text{Arg}(-\sqrt{3} + i) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) + \pi = -\frac{\pi}{6} + \pi = \frac{5\pi}{6}.$$

A different argument for  $-\sqrt{3} + i$  is given by  $\theta = \text{Arg}(-\sqrt{3} + i) + 2\pi = \frac{17\pi}{6}$ . Therefore,

$$-\sqrt{3} + i = 2\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right), \text{ using } \theta = \text{Arg}(-\sqrt{3} + i)$$

$$-\sqrt{3} + i = 2\left(\cos\left(\frac{17\pi}{6}\right) + i\sin\left(\frac{17\pi}{6}\right)\right), \text{ using } \theta \neq \text{Arg}(-\sqrt{3} + i).$$

8.  $-2 - 2\sqrt{3}i$

$$x = -2 \quad y = -2\sqrt{3}$$

$$r = |z| = \sqrt{(-2)^2 + (-2\sqrt{3})^2} = \sqrt{4 + 12} = \sqrt{16} = 4$$

$$\frac{y}{x} = \frac{-2\sqrt{3}}{-2} = \sqrt{3} \quad \theta = \tan^{-1}(\sqrt{3}) = \frac{4\pi}{3}$$

$$z = r(\cos \theta + i \sin \theta) = 4\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right)$$

10.  $\frac{12}{\sqrt{3} + i} = \frac{12}{\sqrt{3} + i} \cdot \frac{\sqrt{3} - i}{\sqrt{3} - i} = \frac{12\sqrt{3} - 12i}{3 - i^2} = \frac{12\sqrt{3} - 12i}{4} = 3\sqrt{3} - 3i$

$$x = 3\sqrt{3} \quad y = -3$$

$$r = |z| = \sqrt{(3\sqrt{3})^2 + (-3)^2} = \sqrt{27 + 9} = \sqrt{36} = 6$$

$$\frac{y}{x} = \frac{-3}{3\sqrt{3}} = \frac{-1}{\sqrt{3}} = -\frac{\sqrt{3}}{3} \quad \theta = \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = -\frac{\pi}{6}$$

$$z = r(\cos \theta + i \sin \theta) = 6\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$