Selected Problem Solutions from Chapter 1

Problem 1.1

(a) First, we find the cumulative distribution function of $\mathbf{y}$

$$\begin{aligned} F_{\mathbf{y}}(y) &= \Pr\{\mathbf{y} \leq y\} \\ &= \Pr\{\max\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n\} \leq y\} \\ &= \Pr\{\mathbf{x}_1 \leq y, \mathbf{x}_2 \leq y, ..., \mathbf{x}_n \leq y\} \\ &= \Pr\{\mathbf{x}_1 \leq y\} \Pr\{\mathbf{x}_2 \leq y\} \cdots \Pr\{\mathbf{x}_n \leq y\} \\ &= F_{\mathbf{x}}^n(y) \end{aligned} \tag{1}$$

The density function is the derivative of this, namely

$$p_{\mathbf{y}}(y) = \frac{\partial}{\partial y} F_{\mathbf{y}}(y) = n F_{\mathbf{x}}^{n-1}(y) p_{\mathbf{x}}(y) \tag{2}$$

(b) For $p(x) = e^{-x}, \quad x \geq 0$, it follows that

$$F(x) = \int_0^x e^{-u} du = 1 - e^{-x} \tag{3}$$

and

$$p(y) = n(1 - e^{-y})^{n-1} e^{-y} \tag{4}$$

Problem 1.3

(a) We follow a number of steps in finding $p(x, y)$ in terms of $p(u_1, u_2)$. First, we find the Jacobean, defined as the determinant of the matrix

$$\begin{bmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial x}{\partial u_2} \\ \frac{\partial y}{\partial u_1} & \frac{\partial y}{\partial u_2} \end{bmatrix}.$$

Substituting in the appropriate terms and simplifying, we find that the determinant of this matrix is

$$-\frac{2\pi\sigma^2}{u_1}.$$

We next find the solutions of u_1 and u_2 in terms of x and y. It is straightforward to show that these solutions are

$$u_1 = \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right) = g(x, y) \tag{5}$$

and

$$u_2 = \frac{1}{2\pi} \tan^{-1}\left(\frac{y}{x}\right) \tag{6}$$

Finally, by definition,

$$p(x, y) = \frac{u_1}{2\pi\sigma^2} p(u_1, u_2)\Big|_{u_1 = g(x,y)} \tag{7}$$

which is

$$p(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right) \tag{8}$$

or a two-dimensional Gaussian density function.

(b) It is clear from Eq. 8 that $p(x, y)$ can be written as

$$p(x,y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{y^2}{2\sigma^2}\right) = p(x)p(y). \tag{9}$$

That is, this transformation produces two independent Gaussian random variables.

Problem 1.5

In general, of course, the k^{th} moment of $\mathbf{n}$ is defined to be

$$E\{\mathbf{n}^k\} = \int_0^\infty n^k p(n) dn = \int_0^\infty \frac{(\theta-1)\gamma^{\theta-1} n^{k+1}}{[n^2+\gamma^2]^{(\theta+1)/2}} dn \tag{10}$$

If we try to solve this equation by substitution[1] of the variable n with $\gamma \tan(\alpha)$, with the corresponding $dn = \gamma \sec^2(\alpha)$, then Eq. 10 becomes

$$E\{\mathbf{n}^k\} = \int_0^{\pi/2} \frac{(\theta-1)\gamma^{\theta-1}\gamma^{k+1}\tan^{k+1}(\alpha)}{[\gamma^2\tan^2(\alpha)+\gamma^2]^{(\theta+1)/2}} \gamma \sec^2(\alpha) d\alpha \tag{11}$$

For $\theta = 4$ and $k = 1$, the mean is

$$E\{\mathbf{n}\} = 3\gamma \int_0^{\theta/2} \sin^2(\alpha)\cos(\alpha) d\alpha = \gamma \quad (\text{for } \theta = 4) \tag{12}$$

and the second moment is

$$E\{\mathbf{n}^2\} = 3\gamma^2 \int_0^{\pi/2} \sin^3(\alpha) d\alpha = 2\gamma^2 \quad (\text{for } \theta = 4) \tag{13}$$

It follows that the variance is

$$V\{\mathbf{n}\} = \gamma^2 \quad (\text{for } \theta = 4) \tag{14}$$

The corresponding results for $\theta = 5$ are

$$E\{\mathbf{n}\} = 4\gamma \int_0^{\pi/2} \sin^2(\alpha)\cos^2(\alpha) d\alpha = \frac{\pi\gamma}{4} \quad (\text{for } \theta = 5), \tag{15}$$

$$E\{\mathbf{n}^2\} = 4\gamma^2 \int_0^{\pi/2} \sin^3(\alpha)\cos(\alpha) d\alpha = \gamma^2 \quad (\text{for } \theta = 5), \tag{16}$$

and

$$V\{\mathbf{n}\} = \left(1 - \frac{\pi^2}{16}\right)\gamma^2 \quad (\text{for } \theta = 5). \tag{17}$$

Problem 1.6

Clearly,

$$E\{[\mathbf{y} - g(\mathbf{x})]^2\} = E\{\mathbf{y}^2\} - 2E\{g(\mathbf{x})\mathbf{y}\} + E\{g^2(\mathbf{x})\} \tag{18}$$

We must calculate the second term. Consider the conditional expectation $E\{g(\mathbf{x})\mathbf{y}|x\}$. This can be written as

$$E\{g(x)\mathbf{y}|x\} = g(x)E\{\mathbf{y}|x\}$$

[1] This is a change of variable to solve an integral; not a *statistical* change of variable.

because, conditioned on x, $g(x)$ is a constant. We can then write

$$E\{g(\mathbf{x})\mathbf{y}\} = E\left\{E\{g(\mathbf{x})\mathbf{y}|x\}\right\} = E\left\{g(x)E\{\mathbf{y}|x\}\right\} = E\{g^2(\mathbf{x})\} \tag{19}$$

and the desired result follows from Eq. 18.

Problem 1.7

By definition, the joint pdf of $\mathbf{x}$ and $\mathbf{y}$ is

$$p(x,y) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right) \tag{20}$$

and we have the change of random variable defined by the two equations

$$\mathbf{u} = \mathbf{x}\cos\theta - \mathbf{y}\sin\theta \tag{21}$$

and

$$\mathbf{w} = \mathbf{x}\sin\theta + \mathbf{y}\cos\theta \tag{22}$$

It follows that the Jacobian is unity for any θ. The inverse functions are

$$\mathbf{x} = \mathbf{u}\cos\theta + \mathbf{w}\sin\theta \tag{23}$$

and

$$\mathbf{y} = -\mathbf{u}\sin\theta + \mathbf{w}\cos\theta \tag{24}$$

Substituting these last two equations into the joint pdf of $\mathbf{x}$ and $\mathbf{y}$ results in the joint pdf of $\mathbf{u}$ and $\mathbf{w}$ as

$$p(u,w) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{u^2+w^2}{2\sigma^2}\right) \tag{25}$$

Problem 1.8

Since the mean is zero, the joint characteristic function is

$$\Phi(\omega_1,\omega_2,\omega_3,\omega_4) = \exp\left(-\frac{1}{2}\vec{\omega}^T R\vec{\omega}\right), \tag{26}$$

where $\vec{\omega} = [\omega_1, ..., \omega_4]^T$ is a column vector of variables and R is a 4×4 autocorrelation matrix whose i^{th} row and j^{th} column is μ_{ij}. We know we can find the first moment of the first component by differentiating $\Phi(\vec{\omega})$ by ω_1

$$\frac{\partial}{\partial\omega_1}\Phi(\vec{\omega}) = \Phi(\vec{\omega})\left[2\left(-\frac{1}{2}\right)\vec{m}_1^T\vec{\omega}\right] \tag{27}$$

where $\vec{m}_i^T = [\mu_{i1}, \mu_{i2}, \mu_{i3}, \mu_{i4}]$. To find the second moment such as $E\{\mathbf{x}_1\mathbf{x}_2\}$, we start by taking the second partial derivative

$$\frac{\partial^2}{\partial\omega_1\partial\omega_2}\Phi(\vec{\omega})\cdot(-\jmath)^2$$

Note that

$$\frac{\partial}{\partial\omega_j}[\vec{m}_i^T\vec{\omega}] = \mu_{ij} = \mu_{ji} \tag{28}$$

so that

$$\frac{\partial^2}{\partial\omega_1\partial\omega_2}\Phi(\vec{\omega}) = \frac{\partial}{\omega_2}\left[-\vec{m}_1^T\vec{\omega}\Phi(\vec{\omega})\right]$$

$$= \Phi(\vec{\omega}) \left[-\mu_{12} + \vec{m}_1^T \vec{\omega} \vec{m}_2^T \vec{\omega} \right]. \tag{29}$$

For $\vec{\omega} = 0$, this is μ_{12}. To find the third moment, we take another partial derivative, resulting in

$$(-\jmath)^3 \frac{\partial^3}{\partial\omega_3 \partial\omega_2 \partial\omega_1} \Phi(\vec{\omega}) =$$

$$(-\jmath)^3 \Phi(\vec{\omega}) \left[\vec{m}_1^T \vec{\omega} \mu_{23} + \vec{m}_2^T \vec{\omega} \mu_{13} + \vec{m}_3^T \vec{\omega} \mu_{12} - \vec{m}_1^T \vec{\omega} \vec{m}_2^T \vec{\omega} \vec{m}_3^T \vec{\omega} \right] \tag{30}$$

The fourth moment is

$$\frac{\partial^4}{\partial\omega_1 \partial\omega_2 \partial\omega_3 \partial\omega_4} \Phi(\vec{\omega}) =$$

$$\Phi(\vec{\omega}) \left[\mu_{12}\mu_{34} + \mu_{13}\mu_{24} + \mu_{14}\mu_{23} \right] \tag{31}$$

plus terms which disappear when $\vec{\omega} = 0$. This gives the desired result.