Contents

CHAPTER ONE

Matrix Two-Person Games

1.1 The Basics

Problems

1.1 There are 100 bankers lined up in each of 100 rows. Pick the richest banker in each row. Javier is the poorest of those. Pick the poorest banker in each column. Raoul is the richest of those. Who is richer: Javier or Raoul?

1.1 Answer: Think of this as a 100 × 100 game matrix and we are looking for the upper and lower values except that we are really doing it for the transpose of the matrix.

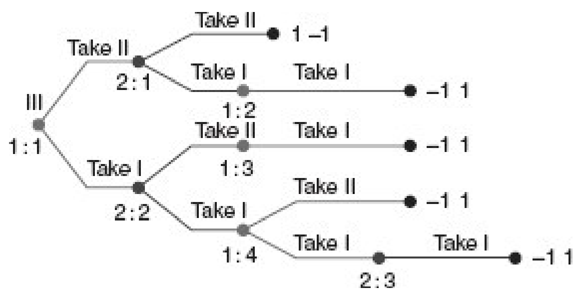
If we take the maximum in each row and then Javier is the minimum maximum, Javier is v^+. If we take the minimum in each column, and Raoul is the maximum of those, then Raoul is the maximum minimum, or v^-. Thus, Javier is richer.

Another way to think of this is that the poorest rich guy is wealthier than the richest poor guy. Common sense.

1.2 In a Nim game start with 4 pennies. Each player may take 1 or 2 pennies from the pile. Suppose player I moves first. The game ends when there are no pennies left and the player who took the last penny pays 1 to the other player.

(a) Draw the game as we did in 2 × 2 Nim.

1.2.a Answer: The game tree is



(b) Write down all the strategies for each player and then the game matrix.

1.2.b Answer:

Using the notation from the figure, we may list the strategies for player I as follows:

1. Go to 2 : 1; if at 1 : 2 take 1. [Same as Take 2. (there are no more choices for I after that)]
2. Go to 2 : 2; if at 1 : 3 take 1; if at 1 : 4 take 2. [Same as Take 1, then if 2 are left, take 1.]
3. Go to 2 : 2; if at 1 : 3 take 1; if at 1 : 4 take 1. [Same as Take 1, then if 2 are left, take 2.]

For player II, the strategies are as follows:

1. If at 2 : 1 take 2; if at 2 : 2 take 2. [Same as If there are 3 left, take 2.]
2. If at 2 : 1 take 2; if at 2 : 2 take 1.
3. If at 2 : 1 take 1; if at 2 : 2 take 2.
4. If at 2 : 1 take 1; if at 2 : 2 take 1; if at 2 : 3 take 1.

The game matrix is

I/II	1	2	3	4
1	1	1	−1	−1
2	−1	−1	−1	−1
3	−1	1	−1	1

(c) Find v^+, v^-. Would you rather be player I or player II?

1.2.c Answer: Since $v^+ = -1$, $v^- = -1$, this game has a value of −1. Player II can always win by playing as follows: If player I takes 2 pennies, then II should take 1. If player I takes 1 penny, then II should take 2 pennies. No matter what player I does, player II wins.

1.3 In the game rock-paper-scissors both players select one of these objects simultaneously. The rules are as follows: paper beats rock, rock beats scissors, and scissors beats paper. The losing player pays the winner $1 after each choice of object. If both choose the same object the payoff is 0.

(a) What is the game matrix?

1.3.a Answer: The rock-paper-scissors game matrix with the rules of the problem is

I/II	Rock	Paper	Scissors
Rock	0	−1	1
Paper	1	0	−1
Scissors	−1	1	0

(b) Find v^+ and v^- and determine whether a saddle point exists in pure strategies, and if so, find it.

1.3.b Answer: $v^+ = 1$, $v^- = -1$. No saddle point in pure strategies since $v^+ > v^-$.

1.4 Each of two players must choose a number between 1 and 5. If a player's choice = opposing player's choice +1, she loses \$2; if a player's choice ≥ opposing player's choice +2, she wins \$1. If both players choose the same number the game is a draw.

(a) What is the game matrix?

1.4.a Answer: The game matrix is

I/II	1	2	3	4	5
1	0	2	−1	−1	−1
2	−2	0	2	−1	−1
3	1	−2	0	2	−1
4	1	1	−2	0	2
5	1	1	1	−2	0

(b) Find v^+ and v^- and determine whether a saddle point exists in pure strategies, and if so, find it.

1.4.b Answer: $v^+ = 1$, $v^- = -1$, no pure saddle point.

1.5 Each player displays either one or two fingers and simultaneously guesses how many fingers the opposing player will show. If both players guess either correctly or incorrectly, the game is a draw. If only one guesses correctly, he wins an amount equal to the total number of fingers shown by both players. Each pure strategy has two components: the number of fingers to show, the number of fingers to guess. Find the game matrix, v^+, v^-, and optimal pure strategies if they exist.

1.5 Answer: For each player we let (i, j) be the pure strategy in which the player shows i fingers, and guesses the other player will show j fingers. The matrix is

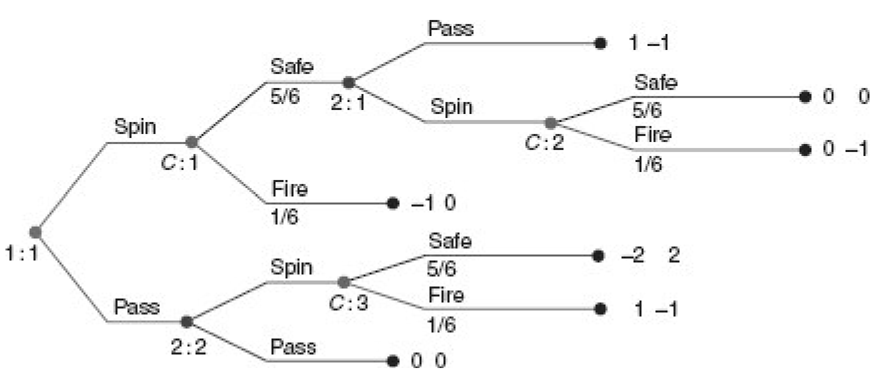
I/II	(1, 1)	(1, 2)	(2, 1)	(2, 2)
(1, 1)	0	2	−3	0
(1, 2)	−2	0	0	3
(2, 1)	3	0	0	−4
(2, 2)	0	−3	4	0

Since $v^+ = 2$, $v^- = -2$, there are no pure optimal strategies.

1.6 In the Russian roulette Example 1.5 suppose that if player I spins and survives and player II decides to pass, then the net gain to I is \$1000 and so I gets all of the additional money that II had to put into the pot in order to pass. Draw the game tree and find the game matrix. What are the upper and lower values? Find the saddle point in pure strategies.

1.6 Answer: The game tree stays the same but the payoff at the end of the Spin-Safe-Pass branch becomes 1, instead of ½.

Here is the game tree:



This is a Gambit generated tree. Any node labeled C is a Chance node. A node labeled with a number indicates that player is making a move. The numbers below the branches in chance moves are the probability that branch is taken. The payoffs at the end of the tree are the payoffs to each player. In a zero sum game, the payoff to player II is the negative of the payoff to player I.

Going through the same calculations as before, we get the game matrix

$$A = \begin{bmatrix} \frac{2}{3} & \frac{2}{3}, & -\frac{1}{36} & -\frac{1}{36} \\ -\frac{3}{2} & 0 & -\frac{3}{2} & 0 \end{bmatrix}.$$

The upper and lower values are $v^- = -\frac{1}{36} = v^+$. There is a pure saddle at row 1, column 3. Even if player I takes the entire pot there will be a saddle at row 1, column 3, both players should spin.

1.7 Let x be an unknown number and consider the matrices

$$A = \begin{bmatrix} 0 & x \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ x & 0 \end{bmatrix}.$$

Show that no matter what x is, each matrix has a pure saddle point.

1.7 Answer: Consider first the game with A. To calculate v^-, we take the minimum of x and 0, written $\min\{x, 0\}$, and the minimum of 1, 2 which is 1. Then

$$v^- = \max\{\min\{x, 0\}, 1\} = 1$$

since $\min\{x, 0\} \le 0$ no matter what x is. Similarly,

$$v^+ = \min\{1, \max\{x, 2\}\} = 1 \text{ since } \max\{x, 2\} \ge 2.$$

Thus, $v^- = v^+ = 1$ and there is a pure saddle at row 2, column 1.

For matrix B, we have $v^- = \max\{1, \min\{x, 0\}\} = 1$, $v^+ = \min\{\max\{2, x\}, 1\} = 1$, and there is a pure saddle at row 1, column 2, no matter what x is.

1.8 If we have a game with matrix A and we modify the game by adding a constant C to every element of A, call the new matrix $A + C$, is it true that $v^+(A + C) = v^+(A) + C$?

1.8 Answer: This is true since

$$v^+(A+C) = \min_{1\le j\le m}\ \max_{1\le i\le n} (a_{ij} + C) = \min_{1\le j\le m}\ \max_{1\le i\le n} a_{ij} + C = v^+(A) + C.$$

It is also true that $v^-(A + C) = v^-(A) + C$.

(a) If it happens that $v^-(A + C) = v^+(A + C)$, will it be true that $v^-(A) = v^+(A)$, and conversely?

1.8.a Answer: It is true that $v^-(A + C) = v^+(A + C) \Leftrightarrow v^-(A) = v^+(A)$. This follows from the first part.

(b) What can you say about the optimal pure strategies for $A + C$ compared to the game for just A?

1.8.b Answer: The previous parts of this problem imply that $A + C$ has a pure saddle point if and only if A has a saddle point. Since

$$a_{ij^*} + C \le a_{i^*j^*} + C \le a_{i^*j} + C \Leftrightarrow a_{ij^*} \le a_{i^*j^*} \le a_{i^*j},$$

we conclude that $A + C$ has a saddle point at (i^*, j^*) if and only if A has a saddle point at (i^*, j^*).

1.9 Consider the square game matrix $A = (a_{ij})$ where $a_{ij} = i - j$ with $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, n$. Show that A has a saddle point in pure strategies. Find them and find $v(A)$.

1.9 Answer: We have

$$v^+(A) = \min_{1\le j\le n}\ \max_{1\le i\le n} (i - j) = \min_{1\le j\le n} (n - j) = n - n = 0$$

and

$$v^-(A) = \max_{1\le i\le n}\ \min_{1\le j\le n} (i - j) = \max_{1\le j\le n} (i - n) = n - n = 0.$$

Thus, $v = 0$ with a pure saddle point at (n, n).

1.10 Player I chooses 1, 2, or 3 and player II guesses which number I has chosen. The payoff to I is |I's number − II's guess|. Find the game matrix. Find v^- and v^+.

1.10 Answer: The game matrix is

I/II	1	2	3
1	0	1	2
2	1	0	1
3	2	1	0

It is easy to see that $v^- = 0$, $v^+ = 1$ and we have no pure saddle point.

1.11 In the Cat versus Rat game, determine v^+ and v^- without actually writing out the matrix. It is a 16 × 16 matrix.

1.11 Answer: $v^+ = 1$, $v^- = 0$ because there is always at least one 1 and one 0 in each row and column.

1.12 In a football game, the offense has two strategies: run or pass. The defense also has two strategies: defend against the run, or defend against the pass. A possible game matrix is

$$A = \begin{bmatrix} 3 & 6 \\ x & 0 \end{bmatrix}.$$

This is the game matrix with the offense as the row player I. The numbers represent the number of yards gained on each play. The first row is run, the second is pass. The first column is defend the run and the second column is defend the pass. Assuming that $x > 0$, find the value of x so that this game has a saddle point in pure strategies.

1.12 Answer: Since $v^- = \max\{3, \min\{x, 0\}\} = 3$ and $v^+ = \min\{\max\{x, 3\}, 6\}$. In order to have a pure saddle, we need $v^+ = 3$ that requires $\max\{x, 3\} = 3$ and so $0 < x \le 3$. Thus, any $0 < x \le 3$ will produce a pure saddle at row 1, column 1.

1.13 Suppose A is a 2×3 matrix and A has a saddle point in pure strategies. Show that it must be true that either one column dominates another, or one row dominates the other, or both. Then find a matrix A that is 3×3 and has a saddle point in pure strategies, but no row dominates another and no column dominates another.

1.13 Answer: Denote the game matrix as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

Without loss of generality, we may as well assume that the saddle is at row 1, column 1, $v^- = v^+ = a_{11}$. Since (1, 1) is a saddle, we have

$$a_{i1} \le a_{11} \le a_{1j},\ i = 2,\ j = 2, 3.$$

In particular, $a_{21} \le a_{11}$. If it is also true that $a_{22} \le a_{12}$ and $a_{23} \le a_{13}$ then row 1 dominates row 2 and we are done. Thus, we need only to suppose that $a_{22} > a_{12}$. Then, from the saddle point inequalities,

$$a_{22} > a_{12} \ge a_{11} \ge a_{21}.$$

But then $a_{12} \ge a_{11}$ and $a_{22} > a_{21}$ says that column 2 is dominated by column 1.

Now consider the 3×3 matrix

$$A = \begin{bmatrix} 4 & -2 & 0 \\ 3 & 1 & 1 \\ 0 & 2 & \frac{1}{2} \end{bmatrix}.$$

Then $v^- = v^+ = 1$ and there is a saddle at row 2, column 3, but no row or column dominates another.

1.2 The von Neumann Minimax Theorem

Problems

1.14 Let $f(x, y) = x^2 + y^2$, $C = D = [-1, 1]$. Find $v^+ = \min_{y\in D} \max_{x\in C} f(x, y)$ and $v^- = \max_{x\in C} \min_{y\in D} f(x, y)$.

1.14 Answer: We have

$$v^+ = \min_{-1\le y\le 1} \max_{-1\le x\le 1} (x^2 + y^2) = \min_{-1\le y\le 1} (1 + y^2) = 1,$$

and

$$v^- = \max_{-1\le x\le 1} \min_{-1\le y\le 1} (x^2 + y^2) = \max_{-1\le x\le 1} (x^2 + 0) = 1.$$

1.15 Let $f(x, y) = y^2 - x^2$, $C = D = [-1, 1]$.

(a) Find $v^+ = \min_{y\in D} \max_{x\in C} f(x, y)$ and $v^- = \max_{x\in C} \min_{y\in D} f(x, y)$.

1.15.a Answer: The upper value is

$$v^+ = \min_{-1\le y\le 1} \max_{-1\le x\le 1} (y^2 - x^2) = \min_{-1\le y\le 1} y^2 = 0,$$

and the lower value is

$$v^- = \max_{-1\le x\le 1} \min_{-1\le y\le 1} (y^2 - x^2) = \max_{-1\le x\le 1} (0) = 0,$$

since if y chooses first, y can choose $y = x$. Thus, $v^+ = 0$, $v^- = 0$.

(b) Show that $(0, 0)$ is a pure saddle point for $f(x, y)$.

1.15.b Answer: We have $f(0, 0) = 0$, and

$$f(x, 0) = -x^2 \le f(0, 0) = 0 \le f(0, y) = y^2, \quad \forall\ -1 \le x, y \le 1.$$

1.16 Let $f(x, y) = (x - y)^2$, $C = D = [-1, 1]$. Find $v^+ = \min_{y\in D} \max_{x\in C} f(x, y)$ and $v^- = \max_{x\in C} \min_{y\in D} f(x, y)$.

1.16 Answer: $v^+ = 1$, $v^- = 0$. Here is why. For $v^- = \max_x \min_y (x - y)^2$, y can be chosen to be $y = x$ to get a minimum of zero. For $v^+ = \min_y \max_x (x - y)^2$, x wants to be as far away from y as possible. So, if $y < 0$, then $x = 1$, and if $y > 0$, then $x = -1$, so

$$\max_{-1\le x\le 1} (x - y)^2 = \begin{cases} (1 + y)^2 & \text{if } y > 0; \\ (1 - y)^2 & \text{if } y \le 0. \end{cases}$$

The minimum of this over $y \in [-1, 1]$ is 1, so $v^+ = 1$. You can see this with the Maple commands

```
 > f:=y->piecewise(y<0, (1-y)^2, y>=0, (1+y)^2);
> plot(f(y),y=-1..1, view= [-1..1,0..3]);
```

Observe that the function $f(x, y)$ is not concave–convex.

1.17 Show that for any matrix $A_{n\times m}$, the function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ defined by $f(x, y) = \sum_{i=1}^n \sum_{j=1}^m a_{ij} x_i y_j = xAy^T$ is convex in $y = (y_1, \ldots, y_m)$ and concave in $x = (x_1, \ldots, x_n)$. In fact, it is **bilinear.**

1.17 Answer: Let $x, \xi \in \mathbb{R}^n$, and $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned} f(\alpha x + \beta\xi, y) &= (\alpha x + \beta\xi)Ay^T \\ &= \alpha(xAy^T) + \beta(\xi Ay^T) \\ &= \alpha f(x, y) + \beta f(\xi, y). \end{aligned}$$

This proves $x \mapsto f(x, y)$ is linear. Similarly, $y \mapsto f(x, y)$ is also linear.

1.18 Show that for any real-valued function $f = f(x, y)$, $x \in C$, $y \in D$, where C and D are any old sets, it is always true that

$$\max_{x\in C} \min_{y\in D} f(x, y) \le \min_{y\in D} \max_{x\in C} f(x, y).$$

1.18 Answer: We have for any $x \in C$,

$$\min_{y\in D} f(x, y) \le f(x, y) \Rightarrow \max_{x\in C} \min_{y\in D} f(x, y) \le \max_{x\in C} f(x, y).$$

The right-hand side is a function of y. The left-hand side is a fixed number, v^-, always below the right-hand side for any y. Thus, the minimum of the right-hand side is

$$\max_{x\in C}\min_{y\in D} f(x,y) = v^- \le \min_{y\in D}\max_{x\in C} f(x,y) = v^+.$$

1.19 Verify that if there is $x^* \in C$ and $y^* \in D$ and a real number v so that

$$f(x^*, y) \ge v,\ \forall y \in D, \ \text{ and } \ f(x, y^*) \le v,\ \forall x \in C,$$

then

$$v = f(x^*, y^*) = \max_{x\in C}\min_{y\in D} f(x,y) = \min_{y\in D}\max_{x\in C} f(x,y).$$

1.19 Answer: Under the assumptions,

$$f(x^*, y) \ge v,\ \forall y \in D, \Rightarrow \min_{y\in D} f(x^*, y) \ge v.$$

Then

$$\max_{x\in C}\min_{y\in D} f(x,y) \ge \min_{y\in D} f(x^*, y) \ge v.$$

Similarly,

$$f(x, y^*) \le v,\ \forall\, x \in C \Rightarrow \max_{x\in C} f(x, y^*) \le v,$$

and then

$$\min_{y\in D}\max_{x\in C} f(x,y) \le \max_{x\in C} f(x, y^*) \le v.$$

Since min max ≥ max min, we have from

$$\min_{y\in D}\max_{x\in C} f(x,y) \le v \le \max_{x\in C}\min_{y\in D} f(x,y)$$

that we must have equality throughout.

1.20 Suppose that $f : [0, 1] \times [0, 1] \to \mathbb{R}$ is strictly concave in $x \in [0, 1]$ and strictly convex in $y \in [0, 1]$ and continuous. Then there is a point (x^*, y^*) so that

$$\min_{y\in[0,1]}\max_{x\in[0,1]} f(x,y) = f(x^*, y^*) = \max_{x\in[0,1]}\min_{y\in[0,1]} f(x,y).$$

In fact, define $y = \phi(x)$ as the function so that $f(x, \phi(x)) = \min_y f(x, y)$. This function is well defined and continuous by the assumptions. Also define the function $x = \psi(y)$ by $f(\psi(y), y) = \max_x f(x, y)$. The new function $g(x) = \psi(\phi(x))$ is then a continuous function taking points in [0, 1] and resulting in points in [0, 1]. There is a theorem, called the **Brouwer fixed-point theorem**, which now guarantees that there is a point $x^* \in [0, 1]$ so that $g(x^*) = x^*$ · Set $y^* = \phi(x^*)$. Verify that (x^*, y^*) satisfies the requirements of a saddle point for f.

1.20 Answer: Use the definitions of $y^* = \phi(x^*)$ and $x^* = \psi(y^*)$. We have

$$f(x^*, y^*) = f(\psi(\varphi(x^*)), \varphi(x^*)) = \max_x f(x, \varphi(x^*)) \ge f(x, \varphi(x^*)) = f(x, y^*),$$

for all $x \in [0, 1]$, and

$$\begin{aligned} f(x^*, y^*) &= f(\psi(\varphi(x^*)), \varphi(x^*)) = \min_y f(\psi(\varphi(x^*)), y) \le f(\psi(\varphi(x^*)), y) \\ &= f(x^*, y), \end{aligned}$$

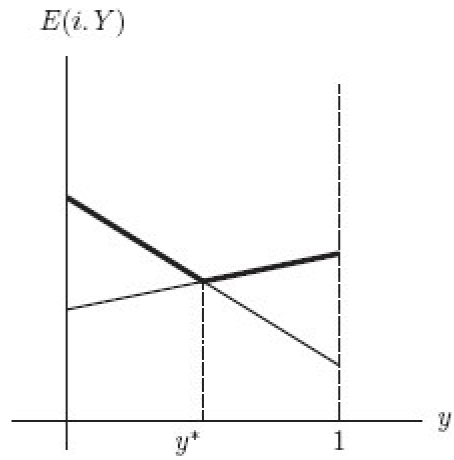
for all $y \in [0, 1]$. Putting these together we have $f(x, y^*) \le f(x^*, y^*) \le f(x^*, y)$ for all $x, y \in [0, 1]$.

1.4 Solving 2 × 2 Games Graphically

Problems

1.21 Following the same procedure as that for player I, look at $E(i, Y)$, $i = 1, 2$ with $Y = (y, 1 - y)$. Graph the lines $E(1, Y) = y + 4(1 - y)$ and $E(2, Y) = 3y + 2(1 - y)$, $0 \le y \le 1$. Now, how does player II analyze the graph to find Y^*?

1.21 Answer: Since player II wants to guarantee that player I gets the smallest maximum, look at the line segments that are the highest and then choose the y^* that gives the smallest maximum payoff. The point of intersection of the two lines is where the y^* will be located and the corresponding horizontal coordinate will be the value of the game.



The result for this problem is $3y + 2(1 - y) = y + 4(1 - y)$ which implies $y^* = ½$ and $v(A) = \frac{5}{2}$.

1.22 Find the value and optimal X^* for the games with matrices

$$\textbf{(a)}\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \qquad \textbf{(b)}\begin{bmatrix} 3 & 1 \\ 5 & 7 \end{bmatrix}$$

What, if anything, goes wrong with (b) if you use the graphical method?

1.22 Answer: For (a), the lines for player I cross where $x - (1 - x) = 2(1 - x)$, which gives $x^* = ¾$. For player II, the lines cross where $y = -y + 2(1 - y)$ which gives $y^* = ½$. Therefore, the solution of the game in mixed strategies is

$$X^* = \left(\frac{3}{4}, \frac{1}{4}\right), \quad Y^* = \left(\frac{1}{2}, \frac{1}{2}\right), \quad value(A) = \frac{1}{2}.$$

For part (b), the matrix has a saddle point at row 2, column 1, so the optimal strategies won't be mixed strategies. If we didn't spot the pure saddle point and applied the graphical method anyway, we would get for player II, the two lines cross where $3y + 1 - y = 5y + 7(1 - y)$, which gives $y^* = \frac{3}{2} > 1$. The second line lies above the first line for the range $0 \le y \le 1$.

1.23 Curly has two safes, one at home and one at the office. The safe at home is a piece of cake to crack and any thief can get into it. The safe at the office is hard to crack and a thief has only a 15% chance of doing it. Curly has to decide where to place his gold bar (worth 1). On the other hand, if the thief hits the wrong place he gets caught (worth −1 to the thief and +1 to Curly). Formulate this as a two-person zero sum matrix game and solve it using the graphical method.

1.23 Answer: Let Curly be the row player and the thief be the column player. The matrix is

I/II	Home	Office
Home	−1	1
Office	1	0.7

The payoff of 0.7 to Curly if he puts the gold bar at the office and the thief hits the office is obtained by calculating $(+1) \times 0.85 + (-1) \times 0.15 = 0.7$. The two lines for the expected payoffs of player I cross where $E((x, 1 - x), 1) = E((x, 1 - x), 2)$, which become $-x + (1 - x) = x + 0.7(1 - x)$. The solution is $x^* = \frac{3}{23}$.

Similarly for player II, the lines cross where $E(1, (y, 1 - y)) = E(2, (y, 1 - y))$, or $-y + (1 - y) = y + 0.7(1 - y)$.

The mixed strategy solution is $X^* = Y^* = \left(\frac{3}{23}, \frac{20}{23}\right), v = \frac{17}{23}$.

1.24 Let z be an unknown number and consider the matrices

$$A = \begin{bmatrix} 0 & z \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 \\ z & 0 \end{bmatrix}.$$

(a) Find $v(A)$ and $v(B)$ for any z.

1.24.a Answer: No matter what z is the lower value is $v^-(A) = \max\{1, \min\{z, 0\}\} = 1$ and the upper value is $v^+(A) = \min\{1, \max\{2, z\}\} = 1$, so there a saddle at row 2, column 1, and $v(A) = 1$.

Similarly, $v^-(B) = \max\{1, \min\{z, 0\}\} = 1$, and $v^+(B) = \min\{\max\{2, z\}, 1\} = 1$, so $v(B) = 1$ and there is a pure saddle at row 1, column 2.

(b) Now consider the game with matrix $A + B$. Find a value of z so that $v(A + B) < v(A) + v(B)$ and a value of z so that $v(A + B) > v(A) + v(B)$. Find the values of $A + B$ using the graphical method. This problem shows that the value is not a linear function of the matrix.

1.24.b Answer: We know $v(A) = v(B) = 1$ and so $v(A) + v(B) = 2$.

Now pick $z = 3$. Then $A + B = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ and the graphical method gives $X^* = Y^* = \left(\frac{1}{2}, \frac{1}{2}\right)$, and $v(A + B) = 3 > v(A) + v(B)$.

Next pick $z = -1$. Then $A + B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, and the graphical method gives $X^* = Y^* = \left(\frac{1}{2}, \frac{1}{2}\right)$, and $v(A + B) = 1 < v(A) + v(B)$.

1.25 Suppose that we have the game matrix

$$A = \begin{bmatrix} 13 & 29 & 8 \\ 18 & 22 & 31 \\ 23 & 22 & 19 \end{bmatrix}.$$

Why can this be reduced to $B = \begin{bmatrix} 18 & 31 \\ 23 & 19 \end{bmatrix}$? Now solve the game graphically.

1.25 Answer: Column 2 may be eliminated by dominance: Any $\frac{9}{13} \leq \lambda \leq \frac{3}{4}$ will make

$$13\lambda + 8(1 - \lambda) \leq 29,$$
$$18\lambda + 31(1 - \lambda) \leq 22,$$
$$23\lambda + 19(1 - \lambda) \leq 22.$$

Once column 2 is gone, row 1 may be dropped. Then we apply the graphical method to get $X^* = \left(0, \frac{4}{17}, \frac{13}{17}\right)$ and $Y^* = \left(\frac{12}{17}, 0, \frac{5}{17}\right)$. The value of the game is $v = \frac{37}{17}$.

1.26 Two brothers, Curly and Shemp, inherit a car worth 8000 dollars. Since only one of them can actually have the car, they agree they will present sealed bids to buy the car from the other brother. The brother that puts in the highest sealed bid gets the car. They must bid in 1000 dollar units. If the bids happen to be the same, then they flip a coin to determine ownership and no money changes hands. Curly can bid only up to 5000, while Shemp can bid up to 8000.

Find the payoff matrix with Curly as the row player and the payoffs the expected net gain (since the car is worth 8000). Find v^-, v^+ and use dominance to solve the game.

1.26 Answer: The matrix is 6 × 9 with the bids 0, 1, …, 5 for Curly and 0, 1, …, 8 for Shemp.

Curly/Shemp	0	1	2	3	4	5	6	7	8
0	4	1	2	3	4	5	6	7	8
1	7	4	2	3	4	5	6	7	8
2	6	6	4	3	4	5	6	7	8
3	5	5	5	4	4	5	6	7	8
4	4	4	4	4	4	5	6	7	8
5	3	3	3	3	3	4	6	7	8

For example, if they bid exactly the same amount, the expected payoff to Curly is ½8000 = 4000. If Curly bids 3 and Shemp bids 6 then Shemp gets the car and pays Curly 6000. Shemp's net gain is 2000. This can be thought of as a constant sum game.

It is immediate that $v^- = v^+ = 4$ and the saddle point is Curly should bid 3000 and Shemp should bid 3000, leading to an expected payoff to Curly of 4000. Of course, Shemp also has an expected payoff of 4000.

1.5 Graphical Solution of 2 × *m* and *n* × 2 Games

Problems

1.27 In the 2 × 2 Nim game, we saw that $v^+ = v^- = -1$. Reduce the game matrix using dominance.

1.27 Answer: The game matrix is

Player I/player II	1	2	3	4	5	6
1	1	1	−1	1	1	−1
2	−1	1	−1	−1	1	−1
3	−1	−1	−1	1	1	1

Column 3 immediately dominates every other column. Then it doesn't matter what row player I chooses because the payoff is always −1.

1.28 Consider the matrix game

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

(a) Find $v(A)$ and the optimal strategies.

1.28.a Answer: The graphical method gives $2x = 2(1 - x) \Rightarrow x^* = ½$. Thus, $v(A) = 1, X^* = Y^* = \left(\frac{1}{2}, \frac{1}{2}\right)$.

(b) Show that $X^* = (\frac{1}{2}, \frac{1}{2}), Y^* = (1, 0)$ is not a saddle point for the game even though it does happen that $E(X^*, Y^*) = v(A)$.

1.28.b Answer: A direct calculation gives $v(A) = 1 = E(X^*, Y^*)$. However, $E(X, Y^*) = 2x$, where $X = (x, 1 - x)$, $0 \le x \le 1$, and it is not true that $2x < v(A) = 1$ for all x in that range. This means $Y^* = (1, 0)$ is not optimal.

1.29 Use the methods of this section to solve the games:

$$\textbf{(a)} \begin{bmatrix} 4 & -3 \\ -9 & 6 \end{bmatrix}, \qquad \textbf{(b)} \begin{bmatrix} 4 & 9 \\ 6 & 2 \end{bmatrix}, \qquad \textbf{(c)} \begin{bmatrix} -3 & -4 \\ -7 & 2 \end{bmatrix}.$$

1.29 Answer: (a) $X^* = \left(\frac{15}{22}, \frac{7}{22}\right)$; **(b)** $Y^* = \left(\frac{7}{9}, \frac{2}{9}\right)$; **(c)** $Y^* = \left(\frac{6}{10}, \frac{4}{10}\right)$.

To see where these come from since they are all 2 × 2 games without pure saddle points, simply find where the two payoff lines cross for each player.

For (a) we must solve $4x - 9(1 - x) = -3x + 6(1 - x) \Rightarrow x = \frac{15}{22}$, and $4y - 3(1 - y) = -9y + 6(1 - y)$ that gives $y = \frac{9}{22}$. Then plugging in to either payoff line, we get $v = -\frac{3}{22}$. The other parts are similar.

1.30 Use (convex) dominance and the graphical method to solve the game with matrix

$$A = \begin{bmatrix} 0 & 5 \\ 1 & 4 \\ 3 & 0 \\ 2 & 2 \end{bmatrix}.$$

1.30 Answer: We may drop row 4 since it is (weakly) dominated by a convex combination of rows 2 and 3. In fact, $2 \le \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 3$, and $2 \le \frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 0$.

The lines corresponding to rows 2 and 3 intersect at $y^* = \frac{2}{3}$. That is, $E(2, Y) = y + 4(1 - y) = E(3, Y) = 3y$ and so $y^* = \frac{2}{3}$. The value of the game is $v = 2$. Since we use only rows 2 and 3, it is easy to calculate from the graph that the saddle point for player I is $X^* = \left(0, \frac{1}{2}, \frac{1}{2}, 0\right)$.