SOLUTIONS TO EXERCISES

CHAPTER 1

1.2 Ordered field axioms.

1.2.0. a) False. Let $a = 2/3$, $b = 1$, $c = -2$, and $d = -1$.
b) False. Let $a = -4$, $b = -1$, and $c = 2$.
c) True. Since $a \le b$ and $b \le a + c$, $|a - b| = b - a \le a + c - a = c$.
d) True. No $a \in \mathbf{R}$ satisfies $a < b - \varepsilon$ for all $\varepsilon > 0$, so the inequality is vacuously satisfied. If you want a more constructive proof, if $b \le 0$ then $a < b - \varepsilon < 0 + 0 = 0$. If $b > 0$, then for $\varepsilon = b$, $a < b - \varepsilon = 0$.

1.2.1. a) If $a < b$ then $a + c < b + c$ by the Additive Property. If $a = b$ then $a + c = b + c$ since $+$ is a function. Thus $a + c \le b + c$ holds for all $a \le b$.
b) If $c = 0$ then $ac = 0 = bc$ so we may suppose $c > 0$. If $a < b$ then $ac < bc$ by the Multiplicative Property. If $a = b$ then $ac = bc$ since $\cdot$ is a function. Thus $ac \le bc$ holds for all $a \le b$ and $c \ge 0$.

1.2.2. a) Suppose $0 \le a < b$ and $0 \le c < d$. Multiplying the first inequality by c and the second by b, we have $0 \le ac \le bc$ and $bc < bd$. Hence by the Transitive Property, $ac < bd$.
b) Suppose $0 \le a < b$. By (7), $0 \le a^2 < b^2$. If $\sqrt{a} \ge \sqrt{b}$ then $a = (\sqrt{a})^2 \ge (\sqrt{b})^2 = b$, a contradiction.
c) If $1/a \le 1/b$, then the Multiplicative Property implies $b = ab(1/a) \le ab(1/b) = a$, a contradiction. If $1/b \le 0$ then $b = b^2(1/b) \le 0$ a contradiction.
d) To show these statements may not hold when $a < 0$, let $a = -2$, $b = -1$, $c = 2$ and $d = 5$. Then $a < b$ and $c < d$ but $ac = -4$ is not less than $bd = -5$, $a^2 = 4$ is not less than $b^2 = 1$, and $1/a = -1/2$ is not less than $1/b = -1$.

1.2.3. a) By definition,

$$a^+ - a^- = \frac{|a| + a}{2} - \left(\frac{|a| - a}{2}\right) = \frac{2a}{2} = a$$

and

$$a^+ + a^- = \frac{|a| + a}{2} + \left(\frac{|a| - a}{2}\right) = \frac{2|a|}{2} = |a|.$$

b) By Definition 1.1, if $a \ge 0$ then $a^+ = (a + a)/2 = a$ and if $a < 0$ then $a^+ = (-a + a)/2 = 0$. Similarly, $a^- = 0$ if $a \ge 0$ and $a^- = -a$ if $a < 0$.

1.2.4. a) $|4x - 2| < 22$ if and only if $-22 < 4x - 2 < 22$ if and only if $-5 < x < 6$.
b) $|1 - 2x| < 7$ if and only if $-7 < 1 - 2x < 7$ if and only if $-8 < -2x < 6$ if and only if $-3 < x < 4$.
c) $|x^3 - x| < x^3$ if and only if $-x^3 < x^3 - x < x^3$ if and only if $-x < 0$ and $2x^3 - x > 0$. The first inequality is equivalent to $x > 0$. Since $2x^3 - x = x(2x^2 - 1)$ implies that $x = 0, \pm 1\sqrt{2}$, the second inequality is equivalent to $-1/\sqrt{2} < x < 0$ or $x > 1/\sqrt{2}$. Therefore, the solution is $(1/\sqrt{2}, \infty)$.
d) We cannot multiply by the denominator $x - 2$ unless we consider its sign.
Case 1: $x - 2 > 0$. Then $2x < 4(x - 2)$ so $8 < 2x$ and $x > 4$.
Case 2: $x - 2 < 0$. Then by the Second Multiplicative Property, $2x > 4(x - 2)$ so $8 > 2x$ and $x < 4$. Thus, the solution is $(-\infty, 2) \cup (4, \infty)$.
e) *Case 1*: $3x^2 - 3 > 0$. Cross multiplying, we obtain $3x^2 < 3x^2 - 3$, i.e., this case is empty.
Case 2: $3x^2 - 3 < 0$. Then by the Second Multiplicative Property, $3x^2 > 3x^2 - 3$, i.e., $0 > -3$. Thus, the solution is $(-1, 1)$.

1.2.5. a) Suppose $a > 2$. Then $a - 1 > 1$ so $1 < \sqrt{a - 1} < a - 1$ by (6). Therefore, $2 < b = 1 + \sqrt{a - 1} < 1 + (a - 1) = a$.
b) Suppose $2 < a < 3$. Then $0 < a - 2 < 1$ so $0 < a - 2 < \sqrt{a - 2} < 1$ by (6). Therefore, $0 < a < 2 + \sqrt{a - 2} = b$.
c) Suppose $0 < a < 1$. Then $0 > -a > -1$, so $0 < 1 - a < 1$. Hence $\sqrt{1 - a}$ is real and by (6), $1 - a < \sqrt{1 - a}$. Therefore, $b = 1 - \sqrt{1 - a} < 1 - (1 - a) = a$.
d) Suppose $3 < a < 5$. Then $1 < a - 2 < 3$ so $1 < \sqrt{a - 2} < a - 2$ by (6). Therefore, $3 < 2 + \sqrt{a - 2} = b < a$.

1.2.6. $a+b-2\sqrt{ab}=(\sqrt{a}-\sqrt{b})^2\geq 0$ for all $a,b\in[0,\infty)$. Thus $2\sqrt{ab}\leq a+b$ and $G(a,b)\leq A(a,b)$. On the other hand, since $0\leq a\leq b$ we have $A(a,b)=(a+b)/2\leq 2b/2=b$ and $G(a,b)=\sqrt{ab}\geq\sqrt{a^2}=a$. Finally, $A(a,b)=G(a,b)$ if and only if $2\sqrt{ab}=a+b$ if and only if $(\sqrt{a}-\sqrt{b})^2=0$ if and only if $\sqrt{a}=\sqrt{b}$ if and only if $a=b$.

1.2.7. a) Since $|x+1|\leq|x|+1$, $|x|\leq 4$ implies $|x^2-1|=|x-1||x+1|\leq 5|x-1|$.

b) Since $|x+6|\leq|x|+6$, $|x|\leq 1$ implies $|x^2+5x-6|=|x+6||x-1|\leq 7|x-1|$.

c) Since $|x+5|\leq|x|+5$, $-3\leq x\leq 3$ implies $|x^2+3x-10|=|x+5||x-2|\leq 8|x-2|$.

d) Since the minimum of x^2+2x-8 on $(-2,0)$ is -9, $-2<x<0$ implies $|x^3+3x^2-6x-8|=|x+1||x^2+2x-8|\leq 9|x+1|<9.5|x+1|$.

1.2.8. a) Since $(1-2n)/(1-4n^2)=1/(1+2n)$, the inequality is equivalent to $1/(1+2n)<0.02=1/50$. Since $1+2n>0$ for all $n\in\mathbf{N}$, it follows that $1+2n>50$, i.e., $n\geq 25$.

b) By factoring, we see that the inequality is equivalent to $(n-1)/n<5/2$, i.e., $2(n-1)<5n$. Thus $-2/3<n$, i.e., the solution is any $n\in\mathbf{N}$.

c) The inequality is equivalent to $n^2+4>1000$. Thus $n>\sqrt{996}\approx 31.56$, i.e., $n\geq 32$.

1.2.9. a) $mn^{-1}+pq^{-1}=mqq^{-1}n^{-1}+pq^{-1}nn^{-1}=(mq+pn)n^{-1}q^{-1}$. But $n^{-1}q^{-1}nq=1$ and uniqueness of multiplicative inverses implies $(nq)^{-1}=n^{-1}q^{-1}$. Therefore, $mn^{-1}+pq^{-1}=(mq+pn)(nq)^{-1}$. Similarly, $mn^{-1}(pq^{-1})=mpn^{-1}q^{-1}=mp(nq)^{-1}$. By what we just proved and (2),

$$\frac{m}{n}+\frac{-m}{n}=\frac{m-m}{n}=\frac{0}{n}=0.$$

Therefore, by the uniqueness of additive inverses, $-(m/n)=(-m)/n$. Similarly, $(m/n)(n/m)=(mn)/(mn)=mn(mn)^{-1}=1$, so $(m/n)^{-1}=n/m$ by the uniqueness of multiplicative inverses.

b) Any subset of $\mathbf{R}$ which contains 0 and 1 will satisfy the Associative and Commutative Properties, the Distributive Law, and have an additive identity 0 and a multiplicative identity 1. By part a), $\mathbf{Q}$ satisfies the Closure Properties, has additive inverses, and every nonzero $q\in\mathbf{Q}$ has a multiplicative inverse. Therefore, $\mathbf{Q}$ satisfies Postulate 1.

c) If $r\in\mathbf{Q}$, $x\in\mathbf{R}\setminus\mathbf{Q}$ but $q:=r+x\in\mathbf{Q}$, then $x=q-r\in\mathbf{Q}$, a contradiction. Similarly, if $rx\in\mathbf{Q}$ and $r\neq 0$, then $x\in\mathbf{Q}$, a contradiction. However, the product of any irrational with 0 is a rational.

d) By the First Multiplicative Property, $mn^{-1}<pq^{-1}$ if and only if $mq=mn^{-1}qn<pq^{-1}nq=np$.

1.2.10. $0\leq(cb-ad)^2=c^2b^2-2abcd+a^2d^2$ implies $2abcd\leq c^2b^2+a^2d^2$. Adding $a^2b^2+c^2d^2$ to both sides, we conclude that $(ab+cd)^2\leq(a^2+c^2)(b^2+d^2)$.

1.2.11. Let $\mathbf{P}:=\mathbf{R}^+$.

a) Let $x\in\mathbf{R}$. By the Trichotomy Property, either $x>0$, $-x>0$, or $x=0$. Thus $\mathbf{P}$ satisfies i). If $x>0$ and $y>0$, then by the Additive Property, $x+y>0$ and by the First Multiplicative Property, $xy>0$. Thus $\mathbf{P}$ satisfies ii).

b) To prove the Trichotomy Property, suppose $a,b\in\mathbf{R}$. By i), either $a-b\in\mathbf{P}$, $b-a=-(a-b)\in\mathbf{P}$, or $a-b=0$. Thus either $a>b$, $b>a$, or $a=b$.

To prove the Transitive Property, suppose $a<b$ and $b<c$. Then $b-a,c-b\in\mathbf{P}$ and it follows from ii) that $c-a=b-a+c-b\in\mathbf{P}$, i.e., $c>a$.

Since $b-a=(b+c)-(a+c)$, it is clear that the Additive Property holds.

Finally, suppose $a<b$, i.e., $b-a\in\mathbf{P}$. If $c>0$ then $c\in\mathbf{P}$ and it follows from ii) that $bc-ac=(b-a)c\in\mathbf{P}$, i.e., $bc>ac$. If $c<0$ then $-c\in\mathbf{P}$, so $ac-bc=(b-a)(-c)\in\mathbf{P}$, i.e., $ac>bc$.

1.3 The Completeness Axiom.

1.3.0. a) True. If $A\cap B=\emptyset$, then $\sup(A\cap B):=-\infty$ and there is nothing to prove. If $A\cap B\neq\emptyset$, then use the Monotone Property.

b) True. If $x\in A$, then $x\leq\sup A$. Since $\varepsilon>0$, we have $\varepsilon x\leq\varepsilon\sup A$, so the latter is an upper bound of B. It follows that $\sup B\leq\varepsilon\sup A$. On the other hand, if $x\in A$, then $\varepsilon x\in B$, so $\varepsilon x\leq\sup B$, i.e., $\sup B/\varepsilon$ is an upper bound for A. It follows that $\sup A\leq\sup B/\varepsilon$.

c) True. If $x\in A$ and $y\in B$, then $x+y\leq\sup A+\sup B$, so $\sup(A+B)\leq\sup A+\sup B$. If this inequality is strict, then $\sup(A+B)-\sup B<\sup A$, and it follows from the Approximation Property that there is an $a_0\in A$ such that $\sup(A+B)-\sup B<a_0$. This implies that $\sup(A+B)-a_0<\sup B$, so by the Approximation Property again, there is a $b_0\in B$ such that $\sup(A+B)-a_0<b_0$. We conclude that $\sup(A+B)<a_0+b_0$, a contradiction.

d) False. Let $A = B = [0, 1]$. Then $A - B = [-1, 1]$ so $\sup(A - B) = 1 \neq 0 = \sup A - \sup B$.

1.3.1. a) Since $x^2 + 3x - 10 = 0$ implies $x = -5$, 2, $\inf E = -5$, $\sup E = 2$.
b) Since $x^2 + 3x - 10 > x^2$ implies $x > 10/3$, $\inf E = 10/3$, $\sup E = 10$.
c) Since $2p^2 - 4q^2 > 0$ implies $p/q > \sqrt{2}$, $\inf E = \sqrt{2}$, $\sup E = 2$.
d) Since $(-1)^n/n = -1/n$ when n is odd and $1/n$ when n is even, $\inf E = -1$, $\sup E = 1/2$.
e) Since $1/n + (-1)^n/n = 0$ when n is odd and $2/n$ when n is even, $\inf E = 0$, $\sup E = 1$.
f) Since $5 - (-4)^n/2^{2n} = 6$ when n is odd and 4 when n is even, $\inf E = 4$, $\sup E = 6$.

1.3.2. Since $a - 1/n < a + 1/n$, choose $r_n \in \mathbf{Q}$ such that $a - 1/n < r_n < a + 1/n$, i.e., $|a - r_n| < 1/n$.

1.3.3. $a < b$ implies $a - \sqrt{2} < b - \sqrt{2}$. Choose $r \in \mathbf{Q}$ such that $a - \sqrt{2} < r < b - \sqrt{2}$. Then $a < r + \sqrt{2} < b$. By Exercise 1.2.9c, $r + \sqrt{2}$ is irrational. Thus set $\xi = r + \sqrt{2}$.

1.3.4. If m is a lower bound of E then so is any $\widetilde{m} \leq m$. If m and $\widetilde{m}$ are both infima of E then $m \leq \widetilde{m}$ and $\widetilde{m} \leq m$, i.e., $m = \widetilde{m}$.

1.3.5. Suppose that E is a bounded, nonempty subset of $\mathbf{Z}$. Since $-E$ is a bounded, nonempty subset of $\mathbf{Z}$, it has a supremum by the Completeness Axiom, and that supremum belongs to $-E$ by Theorem 1.15. Hence by the Reflection Principle, $\inf E = -\sup(-E) \in -(-E) = E$.

1.3.6. a) Let $\epsilon > 0$ and $m = \inf E$. Since $m + \epsilon$ is not a lower bound of E, there is an $a \in E$ such that $m + \epsilon > a$. Thus $m + \epsilon > a \geq m$ as required.
b) By Theorem 1.14, there is an $a \in E$ such that $\sup(-E) - \epsilon < -a \leq \sup(-E)$. Hence by the Second Multiplicative Property and Theorem 1.20, $\inf E + \epsilon = -(\sup(-E) - \epsilon) > a > -\sup(-E) = \inf E$.

1.3.7. a) Let x be an upper bound of E and $x \in E$. If M is any upper bound of E then $M \geq x$. Hence by definition, x is the supremum of E.
b) The correct statement is: If x is a lower bound of E and $x \in E$ then $x = \inf E$.
PROOF. $-x$ is an upper bound of $-E$ and $-x \in -E$ so $-x = \sup(-E)$. Thus $x = -\sup(-E) = \inf E$.
c) If E is the set of points x_n such that $x_n = 1 - 1/n$ for odd n and $x_n = 1/n$ for even n, then $\sup E = 1$, $\inf E = 0$, but neither 0 nor 1 belong to E.

1.3.8. Since $A \subseteq E$, any upper bound of E is an upper bound of A. Since A is nonempty, it follows from the Completeness Axiom that A has a supremum. Similarly, B has a supremum. Moreover, by the Monotone Property, $\sup A, \sup B \leq \sup E$.

Set $M := \max\{\sup A, \sup B\}$ and observe that M is an upper bound of both A and B. If $M < \sup E$, then there is an $x \in E$ such that $M < x \leq \sup E$. But $x \in E$ implies $x \in A$ or $x \in B$. Thus M is not an upper bound for one of the sets A or B, a contradiction.

1.3.9. By induction, $2^n > n$. Hence by the Archimedean Principle, there is an $n \in \mathbf{N}$ such that $2^n > 1/(b - a)$. Let $E := \{k \in \mathbf{N} : 2^n b \leq k\}$. By the Archimedean Principle, E is nonempty. Hence let m_0 be the least element in E and set $q = (m_0 - 1)/2^n$. Since $b > 0$, $m_0 \geq 1$. Since m_0 is least in E, it follows that $m_0 - 1 < 2^n b$, i.e., $q < b$. On the other hand, $m_0 \in E$ implies $2^n b \leq m_0$, so

$$a = b - (b - a) < \frac{m_0}{2^n} - \frac{1}{2^n} = \frac{m_0 - 1}{2^n} = q.$$

1.3.10. Since $|x_n| \leq M$, the set $E_n = \{x_n, x_{n+1}, \dots\}$ is bounded for each $n \in \mathbf{N}$. Thus $s_n := \sup E_n$ exists and is finite by the Completeness Axiom. Moreover, since $E_{n+1} \subseteq E_n$, it follows from the Monotone Property, $s_n \geq s_{n+1}$ for each $n \in \mathbf{N}$. Thus $s_1 \geq s_2 \geq \dots$.

By the Reflection Principle, it follows that $t_1 \leq t_2 \leq \cdots$.

Or, if you prefer a more direct approach, $\sigma_n := \sup\{-x_n, -x_{n+1}, \dots\}$ satisfies $\sigma_1 \geq \sigma_2 \geq \dots$. Since $t_n = -\sigma_n$ for $n \in \mathbf{N}$, it follows from the Second Multiplicative Property that $t_1 \leq t_2 \leq \dots$.

1.3.11. Let $E = \{n \in \mathbf{Z} : n \leq a\}$. If $a \geq 0$, then $0 \in E$. If $a < 0$, then by the Archimedean Principle, there is an $m \in \mathbf{N}$ such that $m > -a$, i.e., $n := -m \in E$. Thus E is nonempty. Since E is bounded above (by a), it follows from the Completeness Axiom and Theorem 1.15 that $n_0 = \sup E$ exists and belongs to E.

Set $k = n_0 + 1$. Since $k > \sup E$, k cannot belong to E, i.e., $a < k$. On the other hand, since $n_0 \in E$ and $b - a > 1$,

$$k = n_0 + 1 \leq a + 1 < a + (b - a) = b.$$

We conclude that $a < k < b$.

1.4 Mathematical Induction.

1.4.0. a) False. If $a = -b = 1$ and $n = 2$, then $(a+b)^n = 0$ is NOT greater than $b^2 = 1$.

b) False. If $a = -3$, $b = 1$, and $n = 2$, then $(a+b)^n = 4$ is not less than or equal to $b^n = 1$.

c) True. If n is even, then $n-k$ and k are either both odd or both even. If they're both odd, then $a^{n-k}b^k$ is the product of two negative numbers, hence positive. If they're both even, then $a^{n-k}b^k$ is the product of two positive numbers, hence positive. Thus by the Binomial Formula,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k = a^n + na^{n-1}b + \sum_{k=2}^{n} \binom{n}{k} a^{n-k}b^k =: a^n + na^{n-1}b + C.$$

Since C is a sum of positive numbers, the promised inequality follows at once.

d) True. By the Binomial Formula,

$$\frac{1}{2^n} = \left(\frac{1}{a} + \frac{a-2}{2a}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{a^k} \frac{(a-2)^{n-k}}{2^{n-k}a^{n-k}} = \sum_{k=0}^{n} \binom{n}{k} \frac{(a-2)^{n-k}}{a^n 2^{n-k}}.$$

1.4.1. a) By hypothesis, $x_1 > 2$. Suppose $x_n > 2$. Then by Exercise 1.2.5a, $2 < x_{n+1} < x_n$. Thus by induction, $2 < x_{n+1} < x_n$ for all $n \in \mathbf{N}$.

b) By hypothesis, $2 < x_1 < 3$. Suppose $2 < x_n < 3$. Then by Exercise 1.2.5b, $0 < x_n < x_{n+1}$. Thus by induction, $0 < x_n < x_{n+1}$ for all $n \in \mathbf{N}$.

c) By hypothesis, $0 < x_1 < 1$. Suppose $0 < x_n < 1$. Then by Exercise 1.2.5c, $0 < x_{n+1} < x_n$. Thus by induction this inequality holds for all $n \in \mathbf{N}$.

d) By hypothesis, $3 < x_1 < 5$. Suppose $3 < x_n < 5$. Then by Exercise 1.2.5d, $3 < x_{n+1} < x_n$. Thus by induction this inequality holds for all $n \in \mathbf{N}$.

1.4.2. a) $0 = (1-1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k}(-1)^k = \sum_{k=0}^{n} \binom{n}{k}(-1)^k$.

b) $(a+b)^n = a^n + na^{n-1}b + \cdots + b^n \ge a^n + na^{n-1}b$.

c) By b), $(1+1/n)^n \ge 1^n + n1^{n-1}(1/n) = 2$.

d) $2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k}$ so $\sum_{k=1}^{n} \binom{n}{k} = 2^n - 1$. On the other hand $\sum_{k=0}^{n-1} 2^k = 2^n - 1$ by induction.

1.4.3. a) This inequality holds for $n = 3$. If it holds for some $n \ge 3$ then

$$2(n+1) + 1 = 2n + 1 + 2 < 2^n + 2 < 2^n + 2^n = 2^{n+1}.$$

b) The inequality holds for $n = 1$. If it holds for n then

$$n + 1 < 2^n + 1 \le 2^n + n < 2^n + 2^n = 2^{n+1}.$$

c) Now $n^2 \le 2^n + 1$ holds for $n = 1, 2,$ and 3. If it holds for some $n \ge 3$ then by a),

$$(n+1)^2 = n^2 + 2n + 1 < 2^n + 2^n = 2^{n+1} < 2^{n+1} + 1.$$

d) We claim that $3n^2 + 3n + 1 \le 2 \cdot 3^n$ for $n = 3, 4, \ldots$. This inequality holds for $n = 3$. Suppose it holds for some n. Then

$$3(n+1)^2 + 3(n+1) + 1 = 3n^2 + 3n + 1 + 6n + 6 \le 2 \cdot 3^n + 6(n+1).$$

Similarly, induction can be used to establish $6(n+1) \le 4 \cdot 3^n$ for $n \ge 1$. (It holds for $n = 1$, and if it holds for n then $6(n+2) = 6(n+1) + 6 \le 4 \cdot 3^n + 6 < 4 \cdot 3^n + 8 \cdot 3^n = 4 \cdot 3^{n+1}$.) Therefore,

$$3(n+1)^2 + 3(n+1) + 1 \le 2 \cdot 3^n + 6(n+1) \le 2 \cdot 3^n + 4 \cdot 3^n = 2 \cdot 3^{n+1}.$$

Thus the claim holds for all $n \ge 3$.

Now $n^3 \le 3^n$ holds by inspection for $n = 1, 2, 3$. Suppose it holds for some $n \ge 3$. Then

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1 \le 3^n + 2 \cdot 3^n = 3^{n+1}.$$

1.4.4. a) The formula holds for $n = 1$. If it holds for n then

$$\sum_{k=1}^{n+1} k = \frac{n(n+1)}{2} + n + 1 = (n+1)(\frac{n}{2} + 1) = \frac{(n+1)(n+2)}{2}.$$

b) The formula holds for $n = 1$. If it holds for n then

$$\sum_{k=1}^{n+1} k^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{n+1}{6}(n(2n+1) + 6(n+1)) = \frac{(n+1)(n+2)(2n+3)}{6}.$$

c) The formula holds for $n = 1$. If it holds for n then

$$\sum_{k=1}^{n+1} \frac{a-1}{a^k} = 1 - \frac{1}{a^n} + \frac{a-1}{a^{n+1}} = 1 - \frac{1}{a^{n+1}}.$$

d) The formula holds for $n = 1$. If it holds for n then

$$\begin{aligned}\sum_{k=1}^{n+1}(2k-1)^2 = \frac{n(4n^2-1)}{3} + (2n+1)^2 &= \frac{2n+1}{3}(2n^2+5n+3) \\ = \frac{2n+1}{3}(2n+3)(n+1) &= \frac{(n+1)(4n^2+8n+3)}{3} \\ = \frac{(n+1)(4(n+1)^2-1)}{3}.\end{aligned}$$

1.4.5. $0 \le a^n < b^n$ holds for $n = 1$. If it holds for n then by (7), $0 \le a^{n+1} < b^{n+1}$.

By convention, $\sqrt[n]{b} \ge 0$. If $\sqrt[n]{a} < \sqrt[n]{b}$ is false, then $\sqrt[n]{a} \ge \sqrt[n]{b} \ge 0$. Taking the nth power of this inequality, we obtain $a = (\sqrt[n]{a})^n \ge (\sqrt[n]{b})^n = b$, a contradiction.

1.4.6. The result is true for $n = 1$. Suppose it's true for some odd number ≥ 1, i.e., $2^{2n-1} + 3^{2n-1} = 5\ell$ for some $\ell, n \in \mathbf{N}$. Then

$$2^{2n+1} + 3^{2n+1} = 4 \cdot 2^{2n-1} + 9 \cdot 3^{2n-1} = 4 \cdot 5\ell + 5 \cdot 3^{2n-1}$$

is evidently divisible by 5. Thus the result is true by induction.

1.4.7. We first prove that $2n! + 2 \le (n+1)!$ for $n = 2, 3, \dots$. It's true for $n = 2$. Suppose that it's true for some $n \ge 2$. Then by the inductive hypothesis,

$$2(n+1)! + 2 = 2(n+1)n! + 2 = 2n! + 2 + 2n \cdot n! \le (n+1)! + 2n \cdot n!.$$

But $2 < n+1$ so we continue the inequality above by

$$2(n+1)! + 2 < (n+1)! + n \cdot (n+1)! = (n+2) \cdot (n+1)! = (n+2)!$$

as required.

To prove that $2^n \le n! + 2$, notice first that it's true for $n = 1$. If it's true for some $n \ge 1$, then by the inequality already proved,

$$2^{n+1} = 2 \cdot 2^n \le 2(n! + 2) = 2n! + 2 + 2 \le (n+1)! + 2$$

as required.

1.4.8. If $n = 1$ or $n = 2$, the result is trivial. If $n \ge 3$, then by the Binomial Formula,

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} > \binom{n}{3} = \frac{n(n-1)(n-2)}{6}.$$

1.4.9. a) If $m = k^2$, then $\sqrt{m} = k$ by definition. On the other hand, if m is not a perfect square, then by Remark 1.28, $\sqrt{m}$ is irrational. In particular, it cannot be rational.

b) If $\sqrt{n+3}+\sqrt{n} \in \mathbf{Q}$ then $n+3+2\sqrt{n+3}\sqrt{n}+n = (\sqrt{n+3}+\sqrt{n})^2 \in \mathbf{Q}$. Since $\mathbf{Q}$ is closed under subtraction and division, it follows that $\sqrt{n^2+3n} \in \mathbf{Q}$. In particular, $n^2+3n = m^2$ for some $m \in \mathbf{N}$. Now n^2+3n is a perfect square when $n = 1$ but if $n > 1$ then

$$(n+1)^2 = n^2+2n+1 < n^2+2n+n = \mathbf{n^2+3n} =< n^2+4n+4 = (n+2)^2.$$

Therefore, the original expression is rational if and only if $n = 1$.

c) By repeating the steps in b), we see that the original expression is rational if and only if $n(n+7) = n^2+7n = m^2$ for some $m \in \mathbf{N}$. If $n > 9$ then

$$(n+3)^2 = n^2+6n+9 < \mathbf{n^2+7n} < n^2+8n+16 = (n+4)^2.$$

Thus the original expression cannot be rational when $n > 9$. On the other hand, it is easy to check that n^2+7n is not a perfect square for $n = 1, 2, \dots, 8$ but is a perfect square, namely $144 = 12^2$, when $n = 9$. Thus the original expression is rational if and only if $n = 9$.

1.4.10. The result holds for $n = 0$ since $c_0 - b_0 = 1$ and $a_0^2 + b_0^2 = c_0^2$. Suppose that $c_{n-1} - b_{n-1} = 1$ and $a_{n-1}^2 + b_{n-1}^2 = c_{n-1}^2$ hold for some $n \geq 0$. By definition, $c_n - b_n = c_{n-1} - b_{n-1} = 1$, so by induction, this difference is always 1. Moreover, by the Binomial Formula, the inductive hypothesis, and what we just proved,

$$\begin{aligned}
a_n^2 + b_n^2 &= (a_{n-1}+2)^2 + (2a_{n-1}+b_{n-1}+2)^2 \\
&= a_{n-1}^2 + 4a_{n-1} + 4 + (2a_{n-1}+2)^2 + 2b_{n-1}(2a_{n-1}+2) + b_{n-1}^2 \\
&= c_{n-1}^2 + 2(a_{n-1}+2) + (2a_{n-1}+2)^2 + 2(c_{n-1}-1)(2a_{n-1}+2) \\
&= c_{n-1}^2 + (2a_{n-1}+2)^2 + 2c_{n-1}(2a_{n-1}+2) \\
&= (2a_{n-1}+c_{n-1}+2)^2 \equiv c_n^2.
\end{aligned}$$

1.5 Inverse Functions and Images.

1.5.0. a) False. Since $(\sin x)' = \cos x$ is negative on $[\pi/2, 3\pi/2]$, f is 1–1 there, but the domain of $\arcsin x$ is $[-\pi/2, \pi/2]$. Thus here, $f^{-1}(x) = \arcsin(\pi - x)$.

b) True. By elementary set algebra and Theorem 1.37,

$$(f^{-1}(A) \cap f^{-1}(B)) \cup f^{-1}(C) = f^{-1}(A \cap B) \cup f^{-1}(C) \supset f^{-1}(A \cap B) \neq \emptyset.$$

c) False. If $X = [0,2]$, $A = [0,1]$ and $B = \{1\}$, then $B \setminus A = \emptyset$ but $(A \setminus B)^c = [0,1)^c = [1,2]$.

d) False. Let $f(x) = x+1$ for $-1 \leq x \leq 0$ and $f(x) = 2x-1$ for $0 < x \leq 1$. Then f takes $[-1,1]$ onto $[-1,1]$ and $f(0) = 1$, but $f^{-1}(f(0)) = f^{-1}(1) = \{0,1\}$.

1.5.1. a) f is 1-1 since $f'(x) = 2 > 0$ for $x \in \mathbf{R}$. If $y = 2x + 4$, then $x = (y - 4)/2$. Therefore $f^{-1}(x) = (x-4)/2$. By looking at the graph, we see that $f(E) = \mathbf{R}$.

b) f is 1-1 since $f'(x) = e^{-1/x}/x^2 > 0$ for $x \in (0, \infty)$. If $y = e^{-1/x}$, then $\log y = -1/x$, i.e., $x = -1/\log y$. Therefore, $f^{-1}(x) = (-1)/\log x$. By looking at the graph, we see that $f(E) = (0, 1)$.

c) f is 1-1 on $(\pi/2, 3\pi/2)$ because $f'(x) = 2\sec^2 x > 0$ there. The inverse is $f^{-1}(x) = \arctan(x/2)$. By looking at the graph, we see that $f(E) = (-\infty, \infty)$.

d) Since $f'(x) = 2x - 4 < 0$ for $x < -3$, f is 1-1 on $(-\infty, -3]$. Since $y = x^2 - 4x + 1$ is a quadratic in x, we have $x = \left(4 \pm \sqrt{16+4\left(y-1\right)}\right)/2 = 2 \pm \sqrt{y+3}$. But x is negative on $(-\infty, -3]$, so we must use the negative sign. Hence $f^{-1}\left(x\right) = 2 - \sqrt{x+3}$. By looking at the graph, we see that $f(E) = [22, \infty)$.

e) By definition,

$$f\left(x\right) = \begin{cases} x-1 & x < -2 \\ 3x+3 & -2 \leq x \leq -1 \\ x+1 & x > -1. \end{cases}$$

Thus f is strictly increasing, hence 1-1, and

$$f^{-1}\left(x\right) = \begin{cases} x+1 & x < -3 \\ \left(x-3\right)/3 & -3 \leq x \leq 0 \\ x-1 & x > 0. \end{cases}$$

i.e., $f^{-1}(x) = (|x| - |x+3| + 3x)/3$. By looking at the graph, we see that $f(E) = (-\infty, \infty)$.

f) Since $f'(x) = (-x^2 - 2x + 1)/(x^2 + 1)^2$ is never zero on $[-1, 0]$, f is 1-1 on $[-1, 0]$. By the quadratic formula, $y = f(x)$ implies $x = \left(1 \pm \sqrt{1-4y\left(y-1\right)}\right)/2y$. Since $x \in [-1, 0]$ we must take the minus sign. Hence

$$f^{-1}\left(x\right) = \begin{cases} \left(1-\sqrt{1-4x\left(x-1\right)}\right)/\,2x & x \neq 0 \\ 1 & x = 0. \end{cases}$$

By looking at the graph, we see that $f(E) = [0, 1]$.

1.5.2. a) f decreases and $f(-2) = 25$, $f(1) = -5$. Therefore, $f(E) = (-5, 25)$. Since $f(x) = -2$ implies $x = 7/10$ and $f(x) = 1$ implies $x = 4/10$, we also have $f^{-1}(E) = (4/10, 7/10)$.

b) The graph of f is a parabola whose absolute minimum is -2 at $x = 0$ and whose maximum on $(-1, 2]$ is 2 at $x = 2$. Therefore, $f(E) = [-2, 2]$. Since f takes ± 2 to 2, $f^{-1}(E) = [-2, -1) \cup (1, 2]$.

c) The graph of f is a parabola whose absolute maximum is 4 at $x = 2$. Since $f(0) = 0$, it follows that $f(E) = [0, 4]$. Since $4x - x^2 = 0$ implies $x = 0, 4$, we also have $f^{-1}(E) = [0, 4]$.

d) The graph of $x^2 - 4x + 5$ is a parabola whose minimum is 1 at $x = 2$. Since log increases on $(0, \infty)$, $f(2) = \log(1) = 0$, and $f(5) = \log(10)$, it follows that $f(E) = [0, \log(10)]$. Since $5 = \log(x^2 - 4x + 5)$ implies $x = 2 \pm \sqrt{e^5 - 1}$, we also have

$$f^{-1}\left(E\right) = \left[2-\sqrt{e^5-1}, 2\right)\bigcup\left(2, 2+\sqrt{e^5-1}\right].$$

e) Since $\sin x$ is periodic with maximum 1 and minimum -1, $f(E) = [-1, 1]$. Since $\sin x$ is nonnegative when $2k\pi \le x \le (2k + 1)\pi$ for some $k \in \mathbf{Z}$, it follows that

$$f^{-1}\left(E\right) = \bigcup_{k\in\mathbf{Z}}\left[2k\pi, \left(2k+1\right)\pi\right]$$

1.5.3. a) The minimum of x on $[-1, 1]$ is -1 and the maximum of $x + 1$ on $[-1, 1]$ is 2. Thus $\cup_{x\in[-1,\,1]}[x, x + 1] = [-1, 2]$.

b) The maximum of $x - 2$ on $[-2, 2]$ is 0 and the minimum is of $x + 2$ on $[-2, 2]$ is 0. Thus $\cap_{x\in[-2,\,2]}[x - 2, x + 2] = \{0\}$.

c) The maximum of $1/k$ for $k \in \mathbf{N}$ is 1. Thus $\cup_{k\in\mathbf{N}}[0, 1/k) = [0, 1)$.

d) The maximum of $1/k$ for $k \in \mathbf{N}$ is 1 and the minimum is of $k + 1$ for $k \in \mathbf{N}$ is 2. Thus $\cap_{k\in\mathbf{N}}[1/k, k + 1] = [1, 2]$.

e) The minimum of $1/k$ for $k \in \mathbf{N}$ is 0 and $1 \in [1 - 1/k, 1 + 1/k]$ for all $k \in \mathbf{N}$. Thus $\cap_{k\in\mathbf{N}}[1 - 1/k, 1 + 1/k] = \{1\}$.

f) The minimum of $-k$ for $k \in \mathbf{N}$ is $-\infty$ and the maximum of k for $k \in \mathbf{N}$ is ∞. Thus $\cup_{k\in\mathbf{N}}(-k, k) = (-\infty, \infty)$.

1.5.4. Suppose x belongs to the left side of (16), i.e., $x \in X$ and $x \notin \cap_{\alpha\in A} E_\alpha$. By definition, $x \in X$ and $x \notin E_\alpha$ for some $\alpha \in A$. Therefore, $x \in E_\alpha^c$ for some $\alpha \in A$, i.e., x belongs to the right side of (16). These steps are reversible.

1.5.5. a) By definition, $x \in f^{-1}(\cup_{\alpha\in A} E_\alpha)$ if and only if $f(x) \in E_\alpha$ for some $\alpha \in A$ if and only if $x \in \cup_{\alpha\in A} f^{-1}(E_\alpha)$.

b) By definition, $x \in f^{-1}(\cap_{\alpha\in A} E_\alpha)$ if and only if $f(x) \in E_\alpha$ for all $\alpha \in A$ if and only if $x \in \cap_{\alpha\in A} f^{-1}(E_\alpha)$.

c) To show $f(f^{-1}(E)) = E$, let $x \in E$. Since $E \subseteq f(X)$, choose $a \in X$ such that $x = f(a)$. By definition, $a \in f^{-1}(E)$ so $x \equiv f(a) \in f(f^{-1}(E))$. Conversely, if $x \in f(f^{-1}(E))$, then $x = f(a)$ for some $a \in f^{-1}(E)$. By definition, this means $x = f(a)$ and $f(a) \in E$. In particular, $x \in E$.

To show $E \subseteq f^{-1}(f(E))$, let $x \in E$. Then $f(x) \in f(E)$, so by definition, $x \in f^{-1}(f(E))$.

1.5.6. a) Let $C = [0, 1]$ and $B = [-1, 0]$. Then $C \setminus B = \{0\}$ and $f(C) = f(B) = [0, 1]$. Thus $f(C \setminus B) = \{0\} \neq \emptyset = f(C) \setminus f(B)$.

b) Let $E = [0, 1]$. Then $f(E) = [0, 1]$ so $f^{-1}(f(E)) = [-1, 1] \neq [0, 1] = E$.

1.5.7. *a) implies b).* By definition, $f(A \setminus B) \supseteq f(A) \setminus f(B)$ holds whether f is 1–1 or not. To prove the reverse inequality, suppose f is 1–1 and $y \in f(A \setminus B)$. Then $y = f(a)$ for some $a \in A \setminus B$. Since f is 1–1, $a = f^{-1}(\{y\})$. Thus $y \neq f(b)$ for any $b \in B$. In particular, $y \in f(A) \setminus f(B)$.

b) implies c). By definition, $A \subseteq f^{-1}(f(A))$ holds whether f is 1–1 or not. Conversely, suppose $x \in f^{-1}(f(A))$. Then $f(x) \in f(A)$ so $f(x) = f(a)$ for some $a \in A$. If $x \notin A$, then it follows from b) that $f(A) = f(A \setminus \{x\}) = f(A) \setminus f(\{x\})$, i.e., $f(x) \notin f(A)$, a contradiction.

c) implies d). By Theorem 1.37, $f(A \cap B) \subseteq f(A) \cap f(B)$. Conversely, suppose $y \in f(A) \cap f(B)$. Then $y = f(a) = f(b)$ for some $a \in A$ and $b \in B$. If $y \notin f(A \cap B)$ then $a \notin B$ and $b \notin A$. Consequently, $f^{-1}(f(\{a\})) \supseteq \{a, b\} \supset \{a\}$, which contradicts c).

d) implies a). If f is not 1–1 then there exist $a, b \in X$ such that $a \neq b$ and $y := f(a) = f(b)$. Hence by d), $\{y\} = f(\{a\}) \cap f(\{b\}) = \emptyset$, a contradiction.

1.6 Countable and uncountable sets.

1.6.0. a) False. The function $f(x) = x$ for $x \in \mathbf{N}$ and $f(x) = 1$ for $x \in \mathbf{R} \setminus \mathbf{N}$ takes $\mathbf{R}$ onto $\mathbf{N}$, but $\mathbf{R}$ is not at most countable.

b) False. The sets $A_m := \{\frac{k}{n} : k \in \mathbf{N}$ and $-2^m \le k \le 2^m\}$ are finite, hence at most countable. Since the dyadic rationals are the union of the A_m's as m ranges over $\mathbf{N}$, they must be at most countable by Theorem 1.42ii.

c) True. If B were at most countable, then its subset $f(A)$ would be at most countable by Theorem 1.41, i.e., there is a function g which takes $f(A)$ onto $\mathbf{N}$. Hence by Exercise 1.6.5a, $g \circ f$ takes A onto $\mathbf{N}$. It follows from Lemma 1.40 that A is at most countable, a contradiction.

d) False, beguiling as it seems! Let $E_n = \{0, 1, \dots, 9\}$ and define f on $E_1 \times E_2 \times \cdots$ by taking each point $(x_1, x_2, \dots)$ onto the number with decimal expansion $0.x_1x_2\cdots$. Clearly (see the proof of Remark 1.39), f takes E onto $[0, 1]$. Since $[0, 1]$ is uncountable, it follows from 1.6.0c that $E_1 \times E_2 \times \cdots$ is uncountable.

1.6.1. The function $2x - 1$ is 1–1 and takes $\mathbf{N}$ onto $\{1, 3, 5, \dots\}$. Thus this set is countable by definition.

1.6.2. By two applications of Theorem 1.42i, $\mathbf{Q} \times \mathbf{Q}$ is countable, hence $\mathbf{Q}^3 := (\mathbf{Q} \times \mathbf{Q}) \times \mathbf{Q}$ is also countable.

1.6.3. Let g be a function that takes A onto B. If A is at most countable, then by Lemma 1.40 there is a function f which takes $\mathbf{N}$ onto A. It follows (see Exercise 1.6.5a) that $g \circ f$ takes $\mathbf{N}$ onto B. Hence by Lemma 1.40, B is at most countable, a contradiction.

1.6.4. By definition, there is an $n \in \mathbf{N}$ and a 1–1 function ϕ which takes $Z := \{1, 2, \dots, n\}$ onto A. Let $\psi(x) := f(\phi(x))$ for $x \in Z$. Since f and ϕ are 1–1, $\psi(x) = \psi(y)$ implies $\phi(x) = \phi(y)$ implies $x = y$. Moreover, since f and ϕ are onto, given $b \in B$ there is an $a \in A$ such that $f(a) = b$, and an $x \in Z$ such that $\phi(x) = a$, hence $\psi(x) \equiv f(\phi(x)) = f(a) = b$. Thus ψ is 1–1 from Z onto B. By definition, then, B is finite.

1.6.5. a) Repeat the proof in Exercise 1.6.4 without referring to $\mathbf{N}$ and Z.

b) By the definition of B_0, it is clear that f takes A onto B_0. Suppose $f^{-1}(x) = f^{-1}(y)$ for some $x, y \in B_0$. Since f is 1–1 from A onto B_0, it follows from Theorem 1.30 that $x = f(f^{-1}(x)) = f(f^{-1}(y)) = y$. Thus f^{-1} is 1–1 on B_0.

c) If f is 1–1 (respectively, onto), then it follows from part a) that $g \circ f$ is 1–1 (respectively, onto).

Conversely, if $g \circ f$ is 1–1 (respectively, onto), then by parts a) and b), $f \equiv g^{-1} \circ g \circ f$ is 1–1 (respectively, onto).

1.6.6. a) We prove this result by induction on n.

Suppose $n = 1$. Since $\phi : \{1\} \to \{1\}$, it must satisfy $\phi(1) = 1$. In particular, in this case ϕ is both 1–1 and onto and there is nothing to prove.

Suppose that the result holds for some integer $n \ge 1$ and let $\phi : \{1, 2, \dots, n + 1\} \to \{1, 2, \dots, n + 1\}$. Set $k_0 = \phi(n + 1)$ and define ψ by

$$\psi(\ell) = \begin{cases} \ell & \ell < k_0 \\ \ell - 1 & \ell > k_0. \end{cases}$$

The ψ is 1–1 from $\{1, 2, \dots, k_0 - 1, k_0 + 1, \dots, n + 1\}$ onto $\{1, 2, \dots, n\}$.

Suppose ϕ is 1–1 on $\{1, 2, \dots, n + 1\}$. Then ϕ is 1–1 on $\{1, 2, \dots, n\}$, hence $\psi \circ \phi$ is 1–1 from $\{1, 2, \dots, n\}$ into $\{1, 2, \dots, n\}$. It follows from the inductive hypothesis that $\psi \circ \phi$ takes $\{1, 2, \dots, n\}$ onto $\{1, 2, \dots, n\}$. By Exercise 1.6.5, ϕ takes $\{1, 2, \dots, n\}$ onto $\{1, 2, \dots, k_0 - 1, k_0 + 1, \dots, n + 1\}$. Since $\phi(n + 1) = k_0$, we conclude that ϕ takes $\{1, 2, \dots, n + 1\}$ onto $\{1, 2, \dots, n + 1\}$.

Conversely, if ϕ takes $\{1, 2, \dots, n+1\}$ onto $\{1, 2, \dots, n+1\}$, then ϕ takes $\{1, 2, \dots, n\}$ onto $\{1, 2, \dots, k_0 - 1, k_0 + 1, \dots, n + 1\}$, so $\psi \circ \phi$ takes $\{1, 2, \dots, n\}$ onto $\{1, 2, \dots, n\}$. It follows from the inductive hypothesis that $\psi \circ \phi$ is 1–1 on $\{1, 2, \dots, n\}$. Hence by Exercise 1.6.5 and construction, ϕ is 1–1 on $\{1, 2, \dots, n + 1\}$.

b) We may suppose that E is nonempty. Hence by hypothesis, there is an $n \in \mathbf{N}$ and a 1–1 function ϕ from E onto $\{1, 2, \dots, n\}$. Moreover, by Exercise 1.6.5b, the function ϕ^{-1} is 1–1 from $\{1, 2, \dots, n\}$ onto E.

Consider the function $\phi^{-1} \circ f \circ \phi$. Clearly, it takes $\{1, 2, \dots, n\}$ into $\{1, 2, \dots, n\}$. Hence by part a), $\phi^{-1} \circ f \circ \phi$ is 1–1 if and only if it is onto. In particular, it follows from Exercise 1.6.5c that f is 1–1 if and only if f is onto.

1.6.7. a) Let $q = k/j$. If $k = 0$ then $n^q = 1$ is a root of the polynomial $x - 1$. If $k > 0$ then n^q is a root of the polynomial $x^j - n^k$. If $k < 0$ then n^q is a root of the polynomial $n^{-k}x^j - 1$. Thus n^q is algebraic.

b) By Theorem 1.42, there are countably many polynomials with integer coefficients. Each polynomial of degree n has at most n roots. Hence the class of algebraic numbers of degree n is a countable union of finite sets, hence countable.

c) Since any number is either algebraic or transcendental, **R** is the union of the set of algebraic numbers and the set of transcendental numbers. By b), the former set is countable. Therefore, the latter must be uncountable by the argument of Remark 1.43.