Solutions to Chapter 2 exercises

2.1 Let $x \in (X \setminus C) \cap D$. Then $x \in X,\ x \in D,\ x \notin C$. So $x \in D,\ x \notin C$ which gives $x \in D \setminus C$. Hence $(X \setminus C) \cap D \subseteq D \setminus C$.

Conversely, if $x \in D \setminus C$ then $x \notin C$ so $x \in X \setminus C$, and $x \in D$. So $x \in (X \setminus C) \cap D$. Hence $D \setminus C \subseteq (X \setminus C) \cap D$.

Together these prove that $(X \setminus C) \cap D = D \setminus C$.

2.2 Suppose that $x \in A \setminus (V \cap A)$. Then $x \in A$ and $x \notin V \cap A$ so $x \notin V$. Then $x \in A$ and $x \in X \setminus V$ so $x \in A \cap (X \setminus V)$. Hence $A \setminus (V \cap A) \subseteq A \cap (X \setminus V)$.

Conversely suppose $x \in A \cap (X \setminus V)$. Then $x \in A$ and $x \in X \setminus V$ so $x \notin V$, hence $x \notin V \cap A$. This shows that $x \in A \setminus (V \cap A)$. Hence $A \cap (X \setminus V) \subseteq A \setminus (V \cap A)$.

Together these prove that $A \setminus (V \cap A) = A \cap (X \setminus V)$.

2.3 Suppose that $x \in V$. Then $x \in X$ and $x \notin X \setminus V = X \cap U$, so $x \notin U$. So $x \in X \subseteq Y$ and $x \notin U$ so $x \in Y \setminus U$. This gives $x \in X \cap (Y \setminus U)$. Hence $V \subseteq X \cap (Y \setminus U)$.

Conversely suppose that $x \in X \cap (Y \setminus U)$. Then $x \in X$, and $x \notin U$, so $x \notin X \cap U = X \setminus V$. Hence $x \in V$. Hence $X \cap (Y \setminus U) \subseteq V$.

Together these show that $V = X \cap (Y \setminus U)$.

2.4 If $(a, b) \in U \times V$ then $a \in U$ so $(a, b) \in U \times Y$ and $b \in V$ so $(a, b) \in X \times V$. Hence $(a, b) \in (X \times V) \cap (U \times Y)$. So $U \times V \subseteq (X \times V) \cap (U \times Y)$.

Conversely if $(a, b) \in (X \times V) \cap (U \times Y)$, then $b \in V$ and $a \in U$ so $(a, b) \in U \times V$. Hence $(X \times V) \cap (U \times Y) \subseteq U \times V$.

Together these give $U \times V = (X \times V) \cap (U \times Y)$.

2.5 If $(x,\ y) \in (U_1 \times V_1) \cap (U_2 \times V_2)$ then $x \in U_1$ and $x \in U_2$ so $x \in U_1 \cap U_2$, and similarly $y \in V_1 \cap V_2$, so $(x,\ y) \in (U_1 \cap U_2) \times (V_1 \cap V_2)$. This shows that

$$(U_1 \times V_1) \cap (U_2 \times V_2) \subseteq (U_1 \cap U_2) \times (V_1 \cap V_2).$$

Conversely if $x \in (U_1 \cap U_2) \times (V_1 \cap V_2)$ then $x \in U_1,\ x \in U_2,\ y \in V_1,\ y \in V_2$ so $(x,\ y) \in U_1 \times V_1$ and also $(x,\ y) \in U_2 \times V_2$, so $(x,\ y) \in (U_1 \times V_1) \cap (U_2 \times V_2)$. This shows that

$$(U_1 \cap U_2) \times (V_1 \cap V_2) \subseteq (U_1 \times V_1) \cap (U_2 \times V_2).$$

Together these show that $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$.

2.6 If $x \in U \cap V$ then $x \in \bigcup_{i \in I} B_{i1}$ and $x \in \bigcup_{j \in J} B_{j2}$, so for some $i_0 \in I$ and $j_0 \in J$ we have $x \in Bi_0 1$ and $x \in B_{j_0 2}$, so

$$x \in B_{i_0 1} \cap B_{j_0 2} \subseteq \bigcup_{(i,j) \in I \times J} B_{i1} \cap B_{j2}.$$

Hence

$$U \cap V \subseteq \bigcup_{(i,j) \in I \times J} B_{i1} \cap B_{j2}.$$

Conversely, if $x \in \bigcup_{(i,j) \in I \times J} B_{i1} \cap B_{j2}$ then for some $i_0 \in I$ and $j_0 \in J$ we have $x \in B_{i_0 1} \cap B_{j_0 2}$, so $x \in B_{i_0 1} \subseteq U$ and similarly $x \in V$ so $x \in U \cap V$. Hence

$$\bigcup_{(i,j) \in I \times J} B_{i1} \cap B_{j2} \subseteq U \cap V.$$

Together these show that

$$U \cap V = \bigcup_{(i,j) \in I \times J} B_{i1} \cap B_{j2}.$$

2.7 (a) Let the distinct equivalence classes be $\{A_i : i \in I\}$. Each A_i, being an equivalence class, satisfies $A_i \subseteq X$. To see that the distinct equivalence classes are disjoint, suppose that for some $i, j \in I$ and some $x \in X$ we have $x \in A_i \cap A_j$. Then for any $a \in A_i$ we have $a \sim x$ and $x \in A_j$, hence $a \in A_j$. This shows $A_i \subseteq A_j$. Similarly $A_j \subseteq A_i$. But this shows that $A_i = A_j$. Thus distinct equivalence classes are mutually disjoint. Finally, any $x \in X$ is in some equivalence class with respect to $\sim$, so $X \subseteq \bigcup_{i \in I} A_i$. Also, since each A_i is a subset of X we have $\bigcup_{i \in I} A_i \subseteq X$. So $X = \bigcup_{i \in I} A_i$.

(b) We define $x_1 \sim x_2$ iff $x_1, x_2 \in A_i$ for some $i \in I$. This is reflexive since each $x \in X$ is in some A_i so $x \sim x$. It is symmetric since if $x_1 \sim x_2$ then $x_1, x_2 \in A_i$ for some $i \in I$, and then also $x_2, x_1 \in A_i$ so $x_2 \sim x_1$. Finally it is transitive since if $x_1 \sim x_2$ and $x_2 \sim x_3$ then $x_1, x_2 \in A_i$ for some $i \in I$ and $x_2, x_3 \in A_j$ for some $j \in I$. Now $x_2 \in A_i \cap A_j$, and since $A_i \cap A_j = \emptyset$ for $i \neq j$, we must have $i = j$. Hence $x_1, x_3 \in A_i$ and we have $x_1 \sim x_3$ as required for transitivity.

2.8 Let $\sim$ be an equivalence relation on the set X. Then $\mathcal{P}(\sim) = \{A_i : i \in I\}$, where $x_1 \sim x_2$ iff $x_1, x_2 \in A_i$ for some $i \in I$. The equivalence relation $\sim' = \sim(\mathcal{P}(\sim))$ is then defined by $x_1 \sim' x_2$ iff $x_1, x_2 \in A_i$ for some $i \in I$, which says that $\sim' = \sim$, that is $\sim(\mathcal{P}(\sim)) = \sim$.

If we begin with a partition $\mathcal{P} = \{A_i : i \in I\}$, then $\sim(\mathcal{P})$ is the equivalence relation $\sim'$ defined by $x_1 \sim' x_2$ iff $x_1, x_2 \in A_i$ for some $i \in I$, and then clearly $\mathcal{P}(\sim') = \mathcal{P}$. This says that $\mathcal{P}(\sim(\mathcal{P})) = \mathcal{P}$.

Solutions to Chapter 3 exercises

3.1 Suppose that $y \in f(A)$. Then $y = f(a)$ for some $a \in A$. Since $A \subseteq B$, also $a \in B$ so $y = f(a) \in f(B)$. By definition, $f(B) \subseteq Y$. This shows that $f(A) \subseteq f(B) \subseteq Y$.

Suppose that $x \in f^{-1}(C)$. Then $f(x) \in C$, so since $C \subseteq D$ also $f(x) \in D$. Hence $x \in f^{-1}(D)$. By definition $f^{-1}(D) \subseteq X$. This shows that $f^{-1}(C) \subseteq f^{-1}(D) \subseteq X$.

3.2 We see, either from a sketch or arguing analytically, that

$$f([0,\, \pi/2]) = [0,\, 1], \quad f([0,\, \infty)) = [-1,\, 1], \quad f^{-1}([0,\, 1]) = \bigcup_{n\in\mathbb{Z}} [2n\pi,\, (2n+1)\pi],$$

$$f^{-1}([0,\, 1/2]) = \bigcup_{n\in\mathbb{Z}} ([2n\pi,\, (2n+1/3)\pi] \cup [(2n+2/3)\pi, (2n+1)\pi]), \quad f^{-1}([-1,\, 1]) = \mathbb{R}.$$

3.3 First suppose that $x \in (g \circ f)^{-1}(U)$. Then $g(f(x)) = (g \circ f)(x) \in U$. Hence by definition of inverse images, $f(x) \in g^{-1}(U)$, and again by definition $x \in f^{-1}(g^{-1}(U))$. This shows that $(g \circ f)^{-1}(U) \subseteq f^{-1}(g^{-1}(U))$.

Now suppose $x \in f^{-1}(g^{-1}(U))$. Then $f(x) \in g^{-1}(U)$, so $g(f(x)) \in U$, that is $(g \circ f)(x) \in U$, and by definition of inverse images, $x \in (g \circ f)^{-1}(U)$. Hence $f^{-1}(g^{-1}(U)) \subseteq (g \circ f)^{-1}(U)$.

These together show that $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$.

3.4 We see that

$f([0,\, 1]) = \{(x,\, 2x) : x \in [0,\, 1]\}$, which is the straight line segment in $\mathbb{R}^2$ joining the origin to the point with coordinates $(1,\, 2)$.

We see that $(x,\, 2x) \in [0,\, 1] \times [0,\, 1]$ iff $0 \leqslant x \leqslant 1/2$, so $f^{-1}([0,\, 1] \times [0,\, 1]) = [0,\, 1/2]$.

We see that $(x,\, 2x) \in D$ iff $x \in \mathbb{R}$ and $x^2 + (2x)^2 \leqslant 1$, which holds iff $5x^2 \leqslant 1$, so $f^{-1}(D) = [-1/\sqrt{5},\, 1/\sqrt{5}]$.

3.5 We know from Proposition 3.14 in the book that if $f : X \to Y$ is onto and $C \subseteq Y$ then $f(f^{-1}(C)) = C$.

Suppose that $f : X \to Y$ is such that $f(f^{-1}(C)) = C$ for any subset C of Y. For any $y \in Y$ we can put $C = \{y\}$, and get that $f(f^{-1}(y)) = \{y\}$. This tells us that there exists $x \in f^{-1}(y)$ (for which of course $f(x) = y$) so $f^{-1}(y) \neq \emptyset$. This proves that f is onto.

3.6 Let $f : X \to Y$. We know from Proposition 3.14 in the book that $A \subseteq f^{-1}(f(A))$ for any $A \subseteq X$. Suppose that f is injective and let $x \in f^{-1}(f(A))$. Then $f(x) \in f(A)$ so $f(x) = f(a)$ for some $a \in A$. But f is injective so $x = a$. This proves that $f^{-1}(f(A)) \subseteq A$, and together these give $A = f^{-1}(f(A))$.

Now suppose that $A = f^{-1}(f(A))$ for any $A \subseteq X$. For any $x \in X$ take $A = \{x\}$ and we get $\{x\} = f^{-1}(f(x))$. this says that if $f(x') = f(x)$ then $x' = x$, that is f is injective.

3.7 (i) We can have $y \neq y'$ with neither y nor y' in the image of f, so that $f^{-1}(y) = f^{-1}(y') = \emptyset$. For a concrete counterexample, define $f : \{0\} \to \{0, 1, 2\}$ by $f(0) = 0$ and take $y = 1,\ y' = 2$.

(ii) Suppose that $f : X \to Y$ is onto and $y, y' \in Y$ with $y \neq y'$. Then $f^{-1}(y) \neq f^{-1}(y')$; for if $f^{-1}(y) = f^{-1}(y')$, then there exists $x \in f^{-1}(y) = f^{-1}(y')$ since f is onto. This gives the contradiction $y = f(x) = y'$.

3.8 We know from Proposition 3.9 in the book that $f(A) \setminus f(B) \subseteq f(A \setminus B)$ for any subsets A, B of X.

Suppose first that also $f(A \setminus B) \subseteq f(A) \setminus f(B)$. Then if $y \in f(A \setminus B)$ we know that $y \notin f(B)$. Hence $f(A \setminus B) \cap f(B) = \emptyset$.

Conversely suppose that $f(A \setminus B) \cap f(B) = \emptyset$. Let $y \in f(A \setminus B)$. Then $y \notin f(B)$. Also, $y = f(x)$ for some $x \in A \setminus B$. Thus $y \in f(A)$, but $y \notin f(B)$, so $y \in f(A) \setminus f(B)$. This proves that $f(A \setminus B) \subseteq f(A) \setminus f(B)$, and together with the opening remark we have $f(A \setminus B) = f(A) \setminus f(B)$.

If $f(A \setminus B) \cap f(B) \neq \emptyset$, let $y \in f(A \setminus B) \cap f(B)$. Then $y = f(x)$ for some $x \in A \setminus B$ and also $y = f(x')$ for some $x' \in B$, and we have $x' \neq x$, so f is not injective. Hence if f is injective then $f(A \setminus B) \cap f(B) = \emptyset$ and $f(A \setminus B) = f(A) \setminus f(B)$ by the first part of the question.

3.9 (a) Suppose that $y \in f(A) \cap C$. Then $y \in C$, and $y = f(x)$ for some $x \in A$. Then $x \in f^{-1}(C)$, so $x \in A \cap f^{-1}(C)$ and $y = f(x) \in f(A \cap f^{-1}(C))$. Hence $f(A) \cap C \subseteq f(A \cap f^{-1}(C))$.

Conversely suppose $y \in f(A \cap f^{-1}(C))$. Then $y = f(x)$ for some $x \in A \cap f^{-1}(C)$. Then $y \in f(A)$ since $x \in A$ and $y = f(x) \in C$ since $x \in f^{-1}(C)$. Hence $f(A \cap f^{-1}(C)) \subseteq f(A) \cap C$.

Together these show that $f(A) \cap C = f(A \cap f^{-1}(C))$.

(b) We apply (a) with $C = f(B)$. This tells us that $f(A) \cap f(B) = f(A \cap f^{-1}(f(B)))$, so since $f^{-1}(f(B)) = B$ we have $f(A) \cap f(B) = f(A \cap B)$.

3.10 Each $f^{-1}(y)$ for $y \in Y$ is non-empty since f is onto. If $y, y' \in Y$ with $y \neq y'$ then we can see that $f^{-1}(y) \cap f^{-1}(y') = \emptyset$ since if $x \in f^{-1}(y) \cap f^{-1}(y')$ then $y = f(x) = y'$, contradicting the hypothesis. Finally, $\bigcup_{y \in Y} f^{-1}(y) = X$ by Proposition 3.7 in the book, since $\bigcup_{y \in Y} \{y\} = Y$.