CHAPTER 2

PHYSICAL MODELS

2.1 Waves: Electromagnetic and Acoustic

EXERCISE 2.1-1

Optical and Ultrasonic Beams. Determine and compare the angular width θ_0 and the depth of focus $2z_0$ of optical and ultrasonic Gaussian beams of equal waist radius, $W_0 = 1$ mm. The optical beam has wavelength of 1 μm, and is traveling in air. The ultrasonic wave has frequency of 10 MHz and is traveling in a medium for which the velocity of sound is $v = 1000$ m/s. If the power of the beams is 1 mW, what is the peak intensity at a distance $z = z_0$ and at $z = 2z_0$?

Optical: $W_0 = 10^{-3}$, $\lambda = 10^{-6}$, $P = 10^{-3}$
Ultrasonic: $W_0 = 10^{-3}$, $v = 10^3$, $\nu = 10^7$, $\lambda = v/\nu = 10^{-4}$, $P = 10^{-3}$

Angular width:
Optical: $\theta_0 = \lambda/\pi W_0 = 3.18 \times 10^{-4} = 0.318$ mrad
Ultrasonic: $\theta_0 = \lambda/\pi W_0 = 0.038 = 38$ mrad

Depth of focus:
Optical: $2z_0 = \pi W_0^2/\lambda = 3.14$ m
Ultrasonic: $2z_0 = \pi W_0^2/\lambda = 0.0314$ = 3.14 cm.

Peak power for both optical and ultrasonic beams: $I_0 = P/W_0^2 = 10^3$ W/m^2 $= 0.1$ W/cm^2
Peak intensity at a distance $z = z_0$ is $I_0/2 = 0.05$ W/cm^2
Peak intensity at a distance $z = 2z_0$ is $I_0/5 = 0.02$ W/cm^2

EXERCISE 2.1-2

Amplitude of an Electromagnetic Wave. Determine the electric field amplitude (volt/m) and the magnetic field amplitude (amperes/m) of an electromagnetic plane wave of intensity 1 mW/cm^2 in free space and in a non-magnetic medium with $\epsilon = 2\epsilon_o$.

$$I = |E_0|^2/2\eta = \frac{1}{2}\eta\,|H_0|^2$$

In air $\eta = 377\ \Omega$. For $I = 10^{-3} \times 10^4 = 10$ W/m^2, $|E_0| = 87$ Volt/m and $|H_0| = 0.23$ Ampere/m.
In a medium with $\epsilon = 2\epsilon_o$, $\eta = 377/\sqrt{2}\ \Omega$ so that $|E_0| = 73$ Volt/m and $|H_0| = 0.27$ Ampere/m.

EXERCISE 2.1-3

Amplitude of an Acoustic Wave. Determine the pressure amplitude (Pa) and the velocity amplitude (m/s) of an acoustic wave or intensity 10 mW/cm^2 in air and in water.

$I = \eta |U_0|^2/2 = |P_0|^2/2\eta$

$I = 10^2$ W/m^2

In air $\eta = 4 \times 10^{-4}$ M Rayls $= 400$ Rayls (N.s.m^{-3})

Therefore, the pressure and velocity amplitudes are $|P_0| = 283$ Pa and $|U_0| = 0.71$ m/s

In water $\eta = 1.48$ M Rayls $= 1.48 \times 10^6$ Rayls (N.s.m^{-3})

Therefore, the pressure and velocity amplitudes are $|P_0| = 1.7 \times 10^4$ Pa and $|U_0| = 0.012$ m/s

EXERCISE 2.1-4

Spherical Acoustic Wave. A sphere of 10-cm nominal radius pulsates at 100 Hz and radiates a spherical wave into air. If the intensity is 50 mW/m^2 at a distance of 1 m from the center of the sphere, determine the acoustic power radiated, the surface velocity of the sphere, and the pressure amplitude and velocity amplitude of the wave at distance of 50 cm from the center of the sphere.

Since the intensity at $r = 1$ m is $I = 50$ mW/m^2, the radiated acoustic power is $4\pi r^2 I = 63$ W.

Since the intensity is inversely proportional to the square of the distance, the intensity at the surface of the radiating sphere is

$I = 50 \times 10^{-3} \times (100/10)^2 = 5$ W/m^2.

The surface velocity of the sphere is obtained from the relation $I = \eta |U_0|^2/2$, where $\eta = 400$ Rayls is the impedance of air. This gives a velocity amplitude $|U_0| = 0.16$ m/s.

At at $r = 50$ cm the velocity amplitude is 5 times smaller than at $r = 10$ cm, so that $|U_0| = 0.032$ m/s

The corresponding pressure amplitude $|P_0| = \eta |U_0| = 12.6$ Pa

EXERCISE 2.1-5

Acoustic Beam. A circular piston-type sonar transducer of 0.5-m radius is mounted in an “infinite” baffle and radiates 5 kW of acoustic power in water at 10 kHz. Determine the width of the beam at a distance at which the intensity is down by 10 dB, assumed to be in the far zone. What is the axial pressure level in dB re 1 μbar (1 bar = 10^6 Pa) at a distance of 10 m from the face of the transducer?

The frequency is $\nu = 10^4$ Hz and the speed of sound in water is $v = 1480$ m/s so that the wavelength is $\lambda = v/\nu$ = 0.148 m. The aperture diameter is $D = 1$ m.

For an incident intensity I_i, the intensity of the diffracted beam on axis is

$$I_0 = I_i \left(\frac{\pi D^2}{4\lambda d} \right)^2,$$

The ratio I_0/I_i is 10 dB, when

$$\left(\frac{\pi D^2}{4\lambda d}\right)^2 = 0.1,$$

This occurs at a distance $d = 16.8$ m

At this distance, the beam width $W = 1.22\lambda d/D = 3$ m

Since the total power is 5 kW, and the area of the piston is $A = \pi D^2/4$,

the intensity $I_i = 5 \times 10^3/A = 6.366 \times 10^3$

The intensity at a distance $d = 10$ m is

$$I_0 = I_i \left(\frac{\pi D^2}{4\lambda d}\right)^2 = 1.8 \times 10^3$$

The impedance of water is $\eta = 1.48 \times 10^6$ Rayls. The pressure is computed from the relation

$I_0 = P_0^2/2\eta$. This gives, $P_0 = \sqrt{2\eta I_0} = 7.3 \times 10^4$ Pa

2.2 Wave Interaction I

EXERCISE 2.2-1

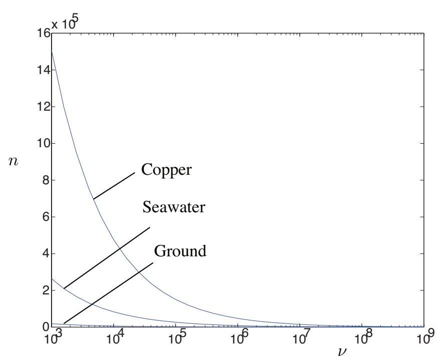
Velocity of an Electromagnetic Wave in a Lossy Medium. Use (2.2-2) and the relation $v = \omega/\beta$ to plot the refractive index $n = c/v$ as a function of frequency for the three materials in Example 2.2-1.

$$\beta = \text{Re}\{\omega\sqrt{\epsilon\mu}\sqrt{1+\sigma/j\omega\epsilon}\}, \qquad n = c/v = c\beta/\omega$$

$$n = c\,\text{Re}\{\sqrt{\epsilon\mu}\sqrt{1+\sigma/j\omega\epsilon}\}, \qquad c = 1/\sqrt{\epsilon_o\mu}$$

$$n = n_0\,\text{Re}\{\sqrt{1+\sigma/j\omega\epsilon}\}, \qquad n_0 = \sqrt{\epsilon_r}$$

$$n = n_0\,\text{Re}\{\sqrt{1-j\nu_c/\nu}\}, \qquad \nu_c = 2\pi\sigma/\epsilon = 2\pi(\sigma/\epsilon_o)\epsilon_r$$



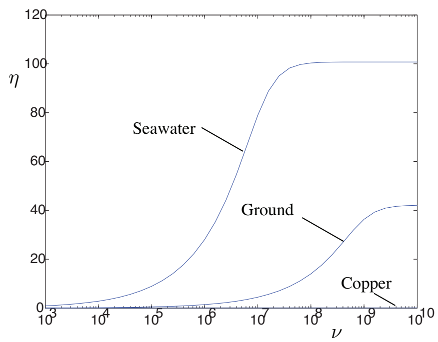
EXERCISE 2.2-2

Impedance of an Electromagnetic Wave in a Lossy Medium. For lossy dielectrics or metals, the permittivity $\epsilon' = \epsilon(1 + \sigma/j\omega\epsilon)$ is complex and the impedance $\eta = \sqrt{\mu_o/\epsilon'}$ is also complex. For sea water with $\epsilon_r = 80$ and $\sigma = 4$, plot the magnitude of the impedance as a function of frequency in the 1 kHz to 10 GHz range.

$$\eta = \sqrt{\mu_o/\epsilon(1 + \sigma/j\omega\epsilon)},$$

$$\eta_0 = \sqrt{\mu_o/\epsilon_0} = 377 \text{ ohms}$$

$$\eta = \eta_0/\sqrt{\epsilon_r(1 - j\nu_c/\nu)}, \qquad \nu_c = \sigma/2\pi\epsilon_r\epsilon_0$$



EXERCISE 2.2-3

Attenuation of an Acoustic Wave in Water. The attenuation coefficient of an ultrasonic wave in water at 1 MHZ corresponds to a half-value thickness of 3.4 m. What is the maximum depth at which a target can be detected using a probe wave producing an intensity of 100 mW/cm^2 if the detector's sensitivity is 1 mW/cm^2? Assume that the transducer is immersed in water so that reflection is not present. Determine the maximum depth for subsurface imaging if the frequency is increased to 2 MHz.

For a medium with attenuation coefficient α, the half-value thickness is $(ln2)/\alpha$

Therefore, a half-value thickness of 3.4 m corresponds to $\alpha = (\ln 2)/3.4 = 0.204$ m^{-1}.

An intensity of 100 mW/cm^2 is reduced to a value 1 mW/cm^2 in a distance d such that

$1/100 = \exp(-\alpha d)$, from which $\alpha d = ln(100)$, or $d = \ln(100)/.204 = 22.6$ m.

For water, $\alpha \propto \omega^a$, where $a = 2$. If the frequency is doubled, the attenuation coefficient is quadrupled, i.e., $\alpha = 0.815$ m^{-1}. In this case the maximum depth is reduced by a factor of 4, from 22.6 m to 4.65 m.

EXERCISE 2.2-4

Reflection and Attenuation of an Electromagnetic Wave. A 1-MHz electromagnetic wave is used to probe a 1-m deep estuary with mixed fresh and seawater. Determine the percentage of power reflected at the water surface and the percentage of power that is reflected from the bottom ground after transmission through water and propagation back into air. The water is a mixture of seawater and fresh water with $\sigma = 0.01$ S/m and $\epsilon_r = 80$, and the bottom ground has $\sigma = 0.01$ S/m and

$\epsilon_r = 14$. Neglect scattering and assume that the water and ground surfaces are perfectly flat. What is the delay time between the first and second echos received in air (resulting from direct reflection at the water surface and reflection at bottom surface)?

Impedance:

$\eta = \sqrt{\mu_o/\epsilon(1+\sigma/j\omega\epsilon)} = \eta_0/\sqrt{\epsilon_r(1-j\nu_c/\nu)}, \quad \nu_c = \sigma/2\pi\epsilon_r\epsilon_0, \quad \eta_0 = \sqrt{\mu_o/\epsilon_0} = 377\ \Omega$

Air: $\eta_a = \eta_0 = 377$

Water: $\eta_w = 3.39 - j3.77$

Ground: $\eta_g = (14.2 - j1.58) \times 10^2$

Reflection at air-water boundary: $R_{aw} = |(\eta_a - \eta_w)/((\eta_a + \eta_w)|^2 = 0.644$

Reflection at water-ground boundary: $R_{wg} = |(\eta_w - \eta_g)/((\eta_w + \eta_g)|^2 = 0.168$

Attenuation coefficient in water: $\alpha = \text{Im}\,\{\omega\sqrt{\epsilon\mu}\sqrt{1+\sigma/j\omega\epsilon}\} = 0.209\ \text{m}^{-1}$

Ratio of power reflected directly from the water surface back into air = $R_{aw} = 0.644$

Ratio of power reflected from bottom back into air

$R = (1-R_{aw})\exp(-\alpha d)R_{wg}\exp(-\alpha d)(1-R_{aw}) = (1-R_{aw})^2 R_{wg}\exp(-2\alpha d)$

With $d = 1$ m, $R = 0.014$.

Velocity in water:

$\beta = \text{Re}\{\omega\sqrt{\epsilon\mu}\sqrt{1+\sigma/j\omega\epsilon}\}, \qquad n = c/v = c\beta/\omega \qquad c = 1/\sqrt{\epsilon_o\mu}$

$n = c\,\text{Re}\{\sqrt{\epsilon\mu}\sqrt{1+\sigma/j\omega\epsilon} = \sqrt{\epsilon_r}\,\text{Re}\{\sqrt{1+\sigma/j\omega\epsilon}\} = 3.77,$

Time delay $= 2d/v = 2nd/c = 25$ ns.

EXERCISE 2.2-5

Reflection and Transmission of an Acoustic Wave at Fluid-Fluid Interface. When an acoustic plane wave is reflected from a fluid-fluid interface it is observed that at normal incidence the pressure amplitude of the reflected wave is 1/2 that of the incident wave. As the angle if incidence is increased, the amplitude of the reflected wave first decreases to zero and then increases until at 30° the reflected wave is a strong as the incident wave. Find the density and sound speed in the second medium if the first medium is water.

At normal incidence, the reflection coefficient is

$$r = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{1}{2}$$

If $\eta_2/\eta_1 = x$, then $(x-1)/(x+1) = 1/2$, from which $x = 3$ or $\eta_2 = 3\eta_1$.

Total reflection occurs at an angle $\sin\theta = v_1/v_2$. Since $\theta_1 = 30°$, $v1/v2 = 1/2$, and $v2 = 2v1$.

For water, $\eta_1 = 1.48 \times 10^6$ and $v_1 = 1480$, so that $\eta_2 = 4.44 \times 10^6$ and $v_2 = 2960$.

The density of the second medium $\rho = \eta_2/v_2 = 4.44 \times 10^6/2960 = 1500$.

EXERCISE 2.2-6

Oblique Transmission of an Acoustic Wave at Air–Water Boundary. A plane acoustic wave of frequency 500 Hz and pressure level 80 dB (re 20 μPa) in air is incident on the calm surface of a lake at an angle θ_1 from normal incidence.

(a) What is the pressure and velocity amplitudes of the incident wave?
(b) What is the maximum value of θ_1 that will still permit propagation into the water?
(c) For $\theta_1 = 10°$, what is the intensity of the sound wave (re 10^{-12} W/m^2) in the water?
(d) How do the answers of a), b), and c) change if the frequency is doubled?

(a) 80 dB is equivalent to a ratio of 10^4, so that the pressure amplitude is

$P_0 = 10^4 \times 20\ \mu$Pa or 0.2 Pa.

In air, the impedance $\eta = 400$ Rayls. The velocity amplitude $U_0 = P_0/\eta = 0.5$ mm/s.

(b) Reflection at the air-water boundary

Air: $\eta_1 = 400, \quad v_1 = 330.$ Water: $\eta_2 = 1.48 \times 10^6, \quad v_2 = 1480.$

The critical angle for total internal reflection is $\theta_c = \sin^{-1}(v_1/v_2) = 0.225$ rad $= 12.9°$.

(c) For $\theta_1 = 10°$, we use the relation $\sin\theta_2 = (v_2/v_1)\sin\theta_1$ to determine $\theta_2 = 51°$.

Using (2.2-16), the reflection coefficient $r = 0.9997$

The transmission coefficient is $t = 1 + r = 1.997$,

so that the pressure amplitude is $1.997 \times 0.2 = 0.3999$ Pa.

The corresponding intensity is $P_0^2/2\eta_2 = 5.4 \times 10^{-8}$ W/m^2,

which is equivalent to $10\log_{10}(5.4 \times 10^{-8}/10^{-12}) = 109$ dB (re 10^{-12} W/m^2),

traveling at an angle $51°$.

(d) If the frequency is doubled, the attenuation coefficient in water is quadrupled and the impedance of water becomes complex.

EXERCISE 2.2-7

Total Transmission of an Electromagnetic Wave at Air–Water Boundary. A plane electromagnetic wave is to be used for under-water imaging. What should be the polarization and the angle of incidence necessary for total transmission at the air-water boundary? Assume that the wavelength is $\lambda = 600$ nm, the absorption coefficient is negligible (see Fig. 2.2-4), and the refractive indexes of air and water are $n = 1$ and 1.333, respectively.

Total transmission occurs at the Brewster angle for waves with parallel polarization.

$\theta_B = \tan^{-1}(n_1/n_2) = \tan^{-1}(1/1.333) = 0.6436$ radians $= 36.88°$

EXERCISE 2.2-8

Impedance Matching Layer for Enhancement of Transmission. You want to maximize the transmission of a plane sound wave from water into aluminum at normal incidence using an impedance matching layer. What must be the optimum characteristic impedance of this layer? If the frequency of the wave is 20 kHz, what must be the density of the material and the speed of sound in it. The velocity and impedance of water and aluminum are listed in Table 2.1-1.

The impedance of water and aluminum are $\eta_1 = 1.48$ and $\eta_3 = 17.33$ M Rayls.

The impedance of matching layer must be $\eta_2 = \sqrt{\eta_1 \eta_3} = 5.06$ M Rayls.

The with of the layer must be a quarter wavelength. If ρ_2 is the density of the medium in the

matching layer, then the velocity is $v_2 = \eta_2/\rho$, and the wavelength is $v_2/\nu = \eta_2/\rho\nu$

The thickness must be $d = \eta_2/4\rho\nu = 5.06 \times 10^6/(4 \times 2 \times 10^4 \rho_2 = 126/\rho_2$.

For example, for a layer of 10-cm width, $\rho_2 = 126/0.1 = 1260$, and

$v_2 = \eta_2/\rho = 5.06 \times 10^6/1260 = 4016$.

EXERCISE 2.2-9

Extinction by Scattering. If the Rayleigh scattering cross section of a single small spherical particle is $\sigma_s = 10^{-10}$ m^2 for light of 800-nm wavelength, determine the total power scattered from the particle when the incident intensity is 1 W/m^2. What is the intensity of the scattered light in the backward direction at a distance of 1 m? Is this polarization dependent? If light travels through a medium with $N = 10^9$ such particles per m^3, determine the extinction coefficient of light traveling through the medium. What is the extinction coefficient if the wavelength is changed to 400 nm?

$\sigma_s = 10^{-10}$ m^2, $\quad I_0 = 1$ W/m^2

$P_s = \sigma_s I_0 = 10^{-10} = 0.1$ nW.

$I_s = (P_s/4\pi r^2)\sin^2\theta$,

For backscattering, $\theta = \pi/2$ so that $I_s = (P_s/4\pi r^2) = 10^{-10}/4\pi \approx 8 \times 10^{-12}$ W/m^2,

This result is independent of polarization.

For a medium with $N = 10^9$ particles per m^3, the extinction coefficient is

$\alpha = N\sigma_s = 10^{-1}$ m -1 at $\lambda = 800$ nm.

Since the cross section is inversely proportional to λ^4, at $\lambda = 400$ nm, the extinction coefficient is larger by a factor of $2^4 = 16$, i.e., $\alpha = 1.6$ m $^{-1}$

EXERCISE 2.2-10

Transmission through Fog. Consider a large patch of fog sufficiently thick that, seen from the air, it is not possible to see the ground. Suppose that the optical properties of this fog bank are: $\mu_s = 10^{-6}$ m^{-1}, $g = 0.8$, and $\mu_a = 7 \times 10^{-4}$ m^{-1}.

(a) Compute the diffuse reflectance $\mathcal{R}$.

(b) Now suppose that embedded in the fog bank is a denser fog layer of height $d = 100$ m, having