Problems

1.1 Show that, for s to be proper complex, its in-phase and quadrature components must be uncorrelated and have the same variance.

Solution

As defined in Appendix C.1.4, proper complexity entails $\mathbb{E}[s^2] = \mathbb{E}[s]^2$. Letting $s = s_\mathrm{i} + \mathrm{j}s_\mathrm{q}$, this corresponds to

$$\mathbb{E}[s_\mathrm{i}^2] - \mathbb{E}[s_\mathrm{q}^2] + 2\mathrm{j}\,\mathbb{E}[s_\mathrm{i}s_\mathrm{q}] = \mathbb{E}[s_\mathrm{i}]^2 - \mathbb{E}[s_\mathrm{q}]^2 + 2\mathrm{j}\,\mathbb{E}[s_\mathrm{i}]\mathbb{E}[s_\mathrm{q}], \tag{1.257}$$

which can be split into twin conditions for the real and imaginary parts, respectively

$$\mathbb{E}[s_\mathrm{i}^2] - \mathbb{E}[s_\mathrm{q}^2] = \mathbb{E}[s_\mathrm{i}]^2 - \mathbb{E}[s_\mathrm{q}]^2 \tag{1.258}$$

and

$$\mathbb{E}[s_\mathrm{i}s_\mathrm{q}] = \mathbb{E}[s_\mathrm{i}]\,\mathbb{E}[s_\mathrm{q}]. \tag{1.259}$$

Condition (1.258) can be rearranged into $\mathrm{var}[s_\mathrm{i}] = \mathrm{var}[s_\mathrm{q}]$ while condition (1.259) amounts to s_i and s_q being uncorrelated.

If $\boldsymbol{s}$ were a vector, then the above conditions would generalize to $\boldsymbol{R}_{s_\mathrm{i}} = \boldsymbol{R}_{s_\mathrm{q}}$ and $\boldsymbol{R}_{s_\mathrm{i}s_\mathrm{q}} = -\boldsymbol{R}^\mathrm{T}_{s_\mathrm{i}s_\mathrm{q}}$. The real and imaginary vectors need not be uncorrelated, only for each entry do the real and imaginary parts need to be uncorrelated (because the diagonal of $\boldsymbol{R}_{s_\mathrm{i}s_\mathrm{q}}$ does need to be zero for the condition $\boldsymbol{R}_{s_\mathrm{i}s_\mathrm{q}} = -\boldsymbol{R}^\mathrm{T}_{s_\mathrm{i}s_\mathrm{q}}$ to hold).

1.2 Let s conform to a 3-PSK constellation defined by $\mathrm{s}_0 = \frac{1}{\sqrt{2}}(1-\mathrm{j})$, $\mathrm{s}_1 = \frac{1}{\sqrt{2}}(-1-\mathrm{j})$, and $\mathrm{s}_2 = \mathrm{j}$. Is this signal proper complex? Is it circularly symmetric?

Solution

This zero-mean ternary signal is not proper complex because $\mathbb{E}[s]^2 = 0$ whereas $\mathbb{E}[s^2] = -1/3$. Referring to Problem 1.1, with $s = s_\mathrm{i} + \mathrm{j}s_\mathrm{q}$, in the ternary signal at hand the in-phase and quadrature components are uncorrelated, but $\mathrm{var}[s_\mathrm{i}] = 1/3$ while $\mathrm{var}[s_\mathrm{q}] = 2/3$.

The signal is not circularly symmetric either because a rotation thereof will result in a constellation with differently positioned points. Only rotations of $\pm 120°$ degrees would yield an equivalent constellation.

1.3 Let s conform to a ternary constellation defined by $\mathrm{s}_0 = -1$, $\mathrm{s}_1 = 0$, and $\mathrm{s}_2 = 1$. Is this signal proper complex? Is it circularly symmetric?

Solution

This zero-mean ternary signal is not proper complex because $\mathbb{E}[s]^2 = 0$ whereas $\mathbb{E}[s^2] = 2/3$. Referring to Problem 1.1, in the ternary signal at hand, $\text{var}[s_\text{i}] = 2/3$ while $\text{var}[s_\text{q}] = 0$.

The signal is not circularly symmetric either because a rotation thereof will result in a constellation with differently positioned points. Only a rotation of $180°$ degrees would yield an equivalent constellation.

1.4 Give an expression for the minimum distance between neighboring points in a one-dimensional constellation featuring M points equidistant along the real axis.

Solution

Such M-PAM constellation (with BPSK as special case for $M = 2$) is described by

$$\left\{(2m+1-M)\,\frac{d_{\text{min}}}{2}\right\} \qquad m = 0,\ldots,M-1, \tag{1.260}$$

from which, by imposing that the variance be unity,

$$d_{\text{min}} = 2\sqrt{\frac{3}{M^2-1}}. \tag{1.261}$$

1.5 Let x be a discrete random variable and let $y = g(x)$ with $g(\cdot)$ an arbitrary function. Is $\mathcal{H}(y)$ larger or smaller than $\mathcal{H}(x)$?

Solution

Since $\mathcal{H}(\cdot)$ quantities the uncertainty in a random variable, $\mathcal{H}(y) \le \mathcal{H}(x)$ with equality if every value of x maps to a distinct value of y, and with strict inequality if multiple values of x map to the same value of y.

1.6 Express the entropy of a discrete random variable x as a function of the information divergence between x and a uniformly distributed counterpart.

Solution

Let $\mathcal{D}$ be the information divergence between x some other another random variable that takes M values equiprobably. Then, from (1.34),

$$\mathcal{D} = \sum_x p_x(\text{x}) \log_2 p_x(\text{x}) - \sum_x p_x(\text{x}) \log_2 \frac{1}{M} \tag{1.262}$$

$$= -\mathcal{H}(x) + \log_2 M. \tag{1.263}$$

It follows that

$$\mathcal{H}(x) = \log_2 M - \mathcal{D}. \tag{1.264}$$

1.7 Express the differential entropy of a real Gaussian variable $x \sim \mathcal{N}(\mu, \sigma^2)$.

Solution

Invoking the PDF in (C.13),

$$\mathfrak{h}(x) = -\mathbb{E}\big[\log_2 f_x(x)\big] \tag{1.265}$$

$$= \mathbb{E}\left[\frac{(x-\mu)^2}{2\sigma^2}\right] \log_2 e - \log_2 \frac{1}{\sqrt{2\pi}\sigma} \tag{1.266}$$

$$= \frac{1}{2}\log_2 e + \frac{1}{2}\log_2(2\pi\sigma^2) \tag{1.267}$$

and thus

$$\mathfrak{h}(x) = \frac{1}{2}\log_2(2\pi e \sigma^2). \tag{1.268}$$

1.8 Compute the differential entropy of a random variable that takes the value 0 with probability $1/3$ and is otherwise uniformly distributed in the interval $[-1, 1]$.

Solution

The differential entropy of a discrete random variable (or, as in this case, a mixed random variable having a discrete component) is $-\infty$. To see that, notice that the density of discrete mass points can be represented by delta functions. Since a delta function can be obtained from a uniform random variable by allowing the support to vanish, we can refer to Example 1.4 and let $b \to 0$, which makes the differential entropy grow unboundedly negative.

As advanced in the text, care must be exercised when dealing with differential entropies.

1.9 Calculate the differential entropy of a random variable x that abides by the exponential distribution

$$f_x(\mathrm{x}) = \frac{1}{\mu} e^{-x/\mu}. \tag{1.269}$$

Solution

Applying the definition of differential entropy,

$$\begin{aligned}
\mathfrak{h} &= -\mathbb{E}[\log_2 f_x(x)] && (1.270)\\
&= \frac{1}{\mu}\int_0^{\infty} e^{-x/\mu}\left(\frac{x}{\mu}+\log_2\mu\right)\mathrm{d}x && (1.271)\\
&= \frac{1}{\mu^2}\int_0^{\infty} x\,e^{-x/\mu}\mathrm{d}x + \frac{\log_2\mu}{\mu}\int_0^{\infty} e^{-x/\mu}\mathrm{d}x && (1.272)\\
&= -\frac{1}{\mu^2}\cdot\mu\,(x+\mu)\,e^{-x/\mu}\Big|_0^{\infty} - \log_2\mu\cdot e^{-x/\mu}\Big|_0^{\infty} && (1.273)\\
&= -\frac{1}{\mu}\cdot(x+\mu)\,e^{-x/\mu}\Big|_0^{\infty} + \log_2\mu && (1.274)\\
&= 1+\log_2\mu. && (1.275)
\end{aligned}$$

1.10 Consider a random variable s such that $\Re\{s\} \sim \mathcal{N}(0,1/2)$ and $\Im\{s\} = q\,\Re(s)$ where $q = \pm 1$ equiprobably. Compute the differential entropy of s, which is complex and Gaussian but not proper, and compare it with that of a standard complex Gaussian.

Solution

Since s is zero-mean and $\mathbb{E}[s^2] = 0$, the variable is indeed proper complex. Its PDF is singular, meaning that its support has zero area on the complex plane; it is supported only on the axes $\Re\{s\} = \Im\{s\}$ and $\Re\{s\} = -\Im\{s\}$. The distribution is thus invariant to rotations of $\pm 90°$ and $180°$, but not to arbitrary rotations, hence it is not circularly symmetric. Moreover, the distribution clearly does not conform to (C.14), indicating that s is not complex Gaussian; precisely, its real and imaginary parts are not jointly Gaussian, even if $s|q=1$ and $s|q=-1$ are Gaussian. The real and imaginary parts of s are also not independent.

Because of the singularity of the PDF, the differential entropy is $\mathfrak{h}(s) = -\infty$. This is the case for any distribution having zero area on the complex plane, something that can be seen by starting with a uniform distribution supported on the rectangle $[-a/2, a/2] \times [-b/2, b/2]$ and letting either a or b vanish. The differential entropy of such uniformly distributed variable equals $\log_2(ab)$, which diverges if either a or b vanish.

Once more we find that the differential entropy is a quantity to be careful with, yet, as emphasized in the text, the mutual information is always well behaved. To see that, consider $y = \sqrt{10}s + z$ with $z \sim \mathcal{N}_{\mathbb{C}}(0,1)$ and let us compute $I(s;y)$. Although not easily tackled analytically, we can compute it numerically, with the side benefit of illustrating the computation of mutual informations numerically by means of histograms. Using the script `mutual_info.m` and the function `chist.m`, from which $I(s;y) = 2.83$ bits. In contrast, for $s' \sim \mathcal{N}_{\mathbb{C}}(0,1)$, we would have $I(s';y) =$

$\log_2(1+10) = 3.46$, confirming that s has a rather reduced information-carrying ability relative to a proper complex Gaussian counterpart because of the strong dependence between the in-phase and quadrature components of s.

1.11 Prove that $\mathfrak{h}(x+a) = \mathfrak{h}(x)$ for any constant a.

Solution

The PDF of $y = x + a$ equals $f_y(\mathrm{y}) = f_x(\mathrm{y} - a)$. Applying the definition of differential entropy,

$$\mathfrak{h}(y) = -\int f_y(\mathrm{y}) \log_2 f_y(\mathrm{y})\,\mathrm{dy} \tag{1.276}$$
$$= -\int f_x(\mathrm{y}-a) \log_2 f_x(\mathrm{y}-a)\,\mathrm{dy} \tag{1.277}$$
$$= -\int f_x(\mathrm{x}) \log_2 f_x(\mathrm{x})\,\mathrm{dx} \tag{1.278}$$
$$= \mathfrak{h}(x) \tag{1.279}$$

where, in (1.278), we applied the change of variables $\mathrm{y} = \mathrm{x} + a$.

1.12 Prove that $\mathfrak{h}(ax) = \mathfrak{h}(x) + \log_2|a|$ for any constant a.

Solution

Consider first the case $a > 0$. Applying (C.11), the PDF of $y = ax$ equals

$$f_y = \frac{f_x(y/a)}{a} \tag{1.280}$$

from which

$$\mathfrak{h}(y) = -\int_{-\infty}^{\infty} f_y(\mathrm{y}) \log_2 f_y(\mathrm{y})\,\mathrm{dy} \tag{1.281}$$
$$= -\frac{1}{a}\int_{-\infty}^{\infty} f_x(\mathrm{y}/a) \log_2 \frac{f_x(\mathrm{y}/a)}{a}\,\mathrm{d}y. \tag{1.282}$$

Applying the change of variables $\mathrm{y} = a\mathrm{x}$,

$$\mathfrak{h}(y) = -\int_{-\infty}^{\infty} f_x(\mathrm{x}) \log_2 f_x(\mathrm{x})\,\mathrm{d}x + \int_{-\infty}^{\infty} f_x(\mathrm{x}) \log_2(a)\,\mathrm{d}x \tag{1.283}$$
$$= \mathfrak{h}(x) + \log_2 a. \tag{1.284}$$

Conversely, for $a < 0$, $f_y = -\frac{f_x(y/a)}{a}$ and

$$\mathfrak{h}(y) = \frac{1}{a}\int_{-\infty}^{\infty} f_x(\mathrm{y}/a) \log_2 \frac{-f_x(\mathrm{y}/a)}{a}\,\mathrm{dy} \tag{1.285}$$

$$= \int_{\infty}^{-\infty} f_x(\mathrm{x}) \log_2 f_x(\mathrm{x})\, \mathrm{d}x - \int_{\infty}^{-\infty} f_x(\mathrm{x}) \log_2(-a)\, \mathrm{d}x \tag{1.286}$$

$$= -\int_{-\infty}^{\infty} f_x(\mathrm{x}) \log_2 f_x(\mathrm{x})\, \mathrm{d}x + \int_{-\infty}^{\infty} f_x(\mathrm{x}) \log_2(-a)\, \mathrm{d}x \tag{1.287}$$

$$= \mathfrak{h}(x) + \log_2(-a). \tag{1.288}$$

Altogether,

$$\mathfrak{h}(x+a) = \mathfrak{h}(x) + \log_2 |a|. \tag{1.289}$$

1.13 Express the differential entropy of the real Gaussian vector $\boldsymbol{x} \sim \mathcal{N}(\mu, \boldsymbol{R})$.

Solution

Using

$$f_{\boldsymbol{x}}(\mathbf{x}) = \frac{1}{\sqrt{\det(2\pi \boldsymbol{R})}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{R}^{-1} (\boldsymbol{x}-\boldsymbol{\mu})} \tag{1.290}$$

we obtain

$$\mathfrak{h}(\boldsymbol{x}) = -\mathbb{E}\big[\log_2 f_{\boldsymbol{x}}(\boldsymbol{x})\big] \tag{1.291}$$

$$= \log_2 \sqrt{\det(2\pi \boldsymbol{R})} + \frac{1}{2}\mathbb{E}\big[(\boldsymbol{x}-\boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{R}^{-1} (\boldsymbol{x}-\boldsymbol{\mu})\big] \log_2 e \tag{1.292}$$

$$= \frac{1}{2}\log_2 \det(2\pi \boldsymbol{R}) + \frac{1}{2}\mathrm{tr}\big(\mathbb{E}\big[(\boldsymbol{x}-\boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{R}^{-1} (\boldsymbol{x}-\boldsymbol{\mu})\big]\big) \log_2 e \tag{1.293}$$

$$= \frac{1}{2}\log_2 \det(2\pi \boldsymbol{R}) + \frac{1}{2}\mathrm{tr}\big(\mathbb{E}\big[\boldsymbol{R}^{-1} (\boldsymbol{x}-\boldsymbol{\mu})(\boldsymbol{x}-\boldsymbol{\mu})^{\mathrm{T}}\big]\big) \log_2 e \tag{1.294}$$

$$= \frac{1}{2}\log_2 \det(2\pi \boldsymbol{R}) + \frac{1}{2}\mathrm{tr}\big(\boldsymbol{R}^{-1}\mathbb{E}\big[(\boldsymbol{x}-\boldsymbol{\mu})(\boldsymbol{x}-\boldsymbol{\mu})^{\mathrm{T}}\big]\big) \log_2 e \tag{1.295}$$

$$= \frac{1}{2}\log_2 \det(2\pi \boldsymbol{R}) + \frac{1}{2}\mathrm{tr}(\boldsymbol{I}) \log_2 e \tag{1.296}$$

$$= \frac{1}{2}\log_2 \det(2\pi e \boldsymbol{R}). \tag{1.297}$$

1.14 Consider the first-order Gauss–Markov process

$$h[n] = \sqrt{1-\varepsilon}\, h[n-1] + \sqrt{\varepsilon}\, w[n] \tag{1.298}$$

where $\{w[n]\}$ is a sequence of IID random variables with $w \sim \mathcal{N}_{\mathbb{C}}(0,1)$.

(a) Express the entropy rate as a function of ε.

(b) Quantify the entropy rate for $\varepsilon = 10^{-3}$.

Note: The Gauss–Markov process underlies a fading model presented in Chapter 3.

Solution

(a) First, let us verify whether $\{h[n]\}$ is stationary. Its expected value at time N satisfies $\mathbb{E}[h[N]] = (1-\varepsilon)^{N/2}\mathbb{E}[h[0]]$ and, since $(1-\epsilon) < 1$, the effect of any nonzero initial value vanishes as $N \to \infty$ and the process becomes zero-mean in steady state. As far as the correlation,

$$E\big[h[n]h^*[n+m]\big] = (1-\varepsilon)^m \qquad \forall n. \tag{1.299}$$

Hence, the mean and correlation are independent of n and the process is wide-sense stationary. And, being Gaussian, it is then (strict-sense) stationary. It follows that we can compute the differential entropy rate as

$$\mathfrak{h} = \mathfrak{h}(h[n]\,|h[n-1]) \tag{1.300}$$
$$= \mathfrak{h}(\sqrt{\varepsilon}w), \tag{1.301}$$

where we have exploited the first-order nature of the process to curtail the conditioning to $h[n-1]$, without the need for earlier values. Recalling Example 1.5, the above gives

$$\mathfrak{h} = \log_2(\pi e \varepsilon). \tag{1.302}$$

(b) For $\varepsilon = 10^{-3}$, $\mathfrak{h} = -6.87$. Its finite value confirms the intuition that the process is regular.

1.15 Verify (1.79) and (1.80).
Hint: Express $\det(\cdot)$ *as the product of the eigenvalues of its argument.*

Solution

Denoting by $\lambda_j(\cdot)$ the jth eigenvalue of a matrix,

$$\log_e \det(\boldsymbol{I} + \rho\boldsymbol{B}) = \log_e \prod_j \lambda_j(\boldsymbol{I} + \rho\boldsymbol{B}) \tag{1.303}$$
$$= \log_e \prod_j (1 + \rho\lambda_j(\boldsymbol{B})) \tag{1.304}$$
$$= \sum_j \log_e(1 + \rho\lambda_j(\boldsymbol{B})), \tag{1.305}$$

from which

$$\frac{\partial}{\partial\rho}\log_e \det(\boldsymbol{I} + \rho\boldsymbol{B}) = \sum_j \frac{\lambda_j(\boldsymbol{B})}{1 + \rho\lambda_j(\boldsymbol{B})} \tag{1.306}$$

and

$$\frac{\partial}{\partial\rho}\log_e \det(\boldsymbol{I} + \rho\boldsymbol{B})\Big|_{\rho=0} = \sum_j \lambda_j(\boldsymbol{B}) \tag{1.307}$$
$$= \mathrm{tr}(\boldsymbol{B}). \tag{1.308}$$

In turn,

$$\frac{\partial^2}{\partial \rho^2} \log_e \det(\boldsymbol{I} + \rho \boldsymbol{B}) = -\sum_j \frac{\lambda_j^2(\boldsymbol{B})}{1 + \rho \lambda_j^2(\boldsymbol{B})} \tag{1.309}$$

and

$$\frac{\partial^2}{\partial \rho^2} \log_e \det(\boldsymbol{I} + \rho \boldsymbol{B}) \bigg|_{\rho=0} = -\sum_j \lambda_j^2(\boldsymbol{B}) \tag{1.310}$$

$$= -\text{tr}(\boldsymbol{B}^2). \tag{1.311}$$

1.16 Show that $I(x_0; x_1; y) \geq I(x_0; y)$ for any random variables x_0, x_1, and y.

Solution

Applying the chain rule of mutual information,

$$I(x_0, x_1; y) = I(x_0; y) + I(x_1; y \,|x_0) \tag{1.312}$$

and, since $I(x_1; y \,|x_0) \geq 0$ (because no mutual information can be strictly negative), it holds that $I(x_0, x_1; y) \geq I(x_0; y)$.

1.17 Let $y = \sqrt{\rho}\,(s_0 + s_1) + z$ where s_0, s_1, and z are independent standard complex Gaussian variables.

(a) Show that $I(s_0, s_1; y) = I(\boldsymbol{s}; \sqrt{\rho}\boldsymbol{A}\boldsymbol{s} + z)$ for $\boldsymbol{s} = [s_0 \; s_1]^{\mathrm{T}}$ and a suitable $\boldsymbol{A}$.

(b) Characterize $I(s_0, s_1; y) - I(s_0; y)$ and approximate its limiting behaviors for $\rho \ll 1$ and $\rho \gg 1$.

(c) Repeat part (b) for the case that s_0 and s_1 are partially correlated. What do you observe?

(d) Repeat part (b) for the modified relationship $y = \sqrt{\rho/2}\,(s_0 + s_1) + z$.

Can you draw any conclusion related to MIMO from this problem?

Solution

(a) Since we want s_0 and s_1 to add up, what we need is $\boldsymbol{A} = [1 \; 1]$. This gives

$$I(\boldsymbol{s}; \sqrt{\rho}\boldsymbol{A}\boldsymbol{s} + z) = I(\boldsymbol{s}; \sqrt{\rho}\,(s_0 + s_1) + z) \tag{1.313}$$

$$= I(s_0, s_1; \sqrt{\rho}\,(s_0 + s_1) + z). \tag{1.314}$$

(b) Applying the chain rule of mutual information,

$$I(s_0, s_1; y) = I(s_0; y) + I(s_1; y \,|s_0) \tag{1.315}$$

where $I(s_0; y)$ corresponds to the transmission of $s_0 \sim \mathcal{N}_{\mathbb{C}}(0, 1)$ over a channel with power gain ρ in the face of noise $z + \sqrt{\rho} s_1 \sim \mathcal{N}_{\mathbb{C}}(0, 1+\rho)$; the SNR is thus $\rho/(1+\rho)$ and, applying Example 1.7, we find that

$$I(s_0; y) = \log_2(1 + \mathsf{SNR}) \tag{1.316}$$
$$= \log_2\left(1 + \frac{\rho}{1+\rho}\right) \tag{1.317}$$
$$= \log_2\left(\frac{1+2\rho}{1+\rho}\right). \tag{1.318}$$

In turn, $I(s_1; y\,|s_0)$ corresponds to the transmission of $s_0 + s_1 \sim \mathcal{N}_{\mathbb{C}}(s_0, 1)$ over a channel with power gain ρ in the face of noise $z \sim \mathcal{N}_{\mathbb{C}}(0, 1)$; the SNR is ρ and s_0 is immaterial, such that

$$I(s_1; y\,|s_0) = \log(1+\rho). \tag{1.319}$$

Altogether, (1.315) equals

$$I(s_0, s_1; y) = \log_2\left(\frac{1+2\rho}{1+\rho}\right) + \log_2(1+\rho) \tag{1.320}$$
$$= \log_2(1+2\rho). \tag{1.321}$$

The above result can be obtained more expeditiously by capitalizing on part (a) of the problem and Example 1.13. Since $\boldsymbol{s} \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{0}, \boldsymbol{I})$,

$$I(\boldsymbol{s}; y) = \log_2 \det(1 + \rho \boldsymbol{A}\boldsymbol{I}\boldsymbol{A}^*) \tag{1.322}$$
$$= \log_2(1 + \rho\,[1\ 1][1\ 1]^{\mathrm{T}}) \tag{1.323}$$
$$= \log_2(1+2\rho). \tag{1.324}$$

For $\rho \ll 1$,

$$I(s_0, s_1; y) \approx 2\rho \log_2 e \tag{1.325}$$

whereas, for $\rho \gg 1$,

$$I(s_0, s_1; y) \approx 1 + \log_2 \rho. \tag{1.326}$$

(c) Letting $R = \mathbb{E}[s_0 s_1^*]$, we have that $\boldsymbol{s} \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{0}, \boldsymbol{R}_{\boldsymbol{s}})$ with

$$\boldsymbol{R}_{\boldsymbol{s}} = \begin{bmatrix} 1 & R \\ R^* & 1 \end{bmatrix} \tag{1.327}$$

and (1.322) generalizes into

$$I(\boldsymbol{s}; y) = \log_2 \det(1 + \rho\, \boldsymbol{A}\boldsymbol{R}_{\boldsymbol{s}}\boldsymbol{A}^*) \tag{1.328}$$
$$= \log_2 \det\big(1 + 2\rho\,(1 + \Re\{R\})\big), \tag{1.329}$$

which is maximized for $R = 1$, i.e., with the same signal transmitted from both antennas. This result is the seed of beamforming: when multiple antennas are available only at the transmitter (MISO configurations), the optimum strategy is to transmit a single signal and ensure its coherent combining at the receiver.

(d) In this case, $I(\boldsymbol{s}; y) = \log_2(1+\rho)$, which is the mutual information that would be achieved if only s_0 or s_1 were transmitted.

The benefits of MISO over SISO emanate from an increase in the receive SNR, either because of mere accumulation if every additional transmit signal contributes its own additional power, or from coherent combining at the receiver. Multiple antennas at both ends would be required to obtain more than a power gain, to obtain a mutual information multiplier.

1.18 Let s be of unit variance and uniformly distributed on a disk while $z \sim \mathcal{N}_{\mathbb{C}}(0,1)$.

(a) What is the first-order expansion of $\mathcal{I}(\rho) = I(s; \sqrt{\rho}s + z)$ for small ρ?

(b) What is the leading term in the expansion of $\mathcal{I}(\rho)$ for large ρ?

Note: The signal distribution in this problem can be interpreted as a dense set of concentric ∞-PSK rings, conveying information in both phase and magnitude.

Solution

(a) The in-phase and quadrature components of s are independent and of equal variance, hence the signal is proper complex. (It is further circularly symmetric.) It follows that its low-ρ expansion abides by (1.52), and the first-order expansion specifically is $\mathcal{I}(\rho) = \rho \log_2 e + \mathcal{O}(\rho^2)$.

(b) As far as the leading term for high ρ is concerned, we can approximate the mutual information in this regime as

$$\mathcal{I}(\rho) = \mathfrak{h}(\sqrt{\rho}s + z) - \mathfrak{h}(z) \tag{1.330}$$

$$\approx \mathfrak{h}(\sqrt{\rho}s) - \mathfrak{h}(z) \tag{1.331}$$

$$= \mathfrak{h}(\sqrt{\rho}s) - \log(\pi e), \tag{1.332}$$

where (1.332) follows from $z \sim \mathcal{N}_{\mathbb{C}}(0,1)$.

For ∞-PSK, $\sqrt{\rho}s$ is uniformly distributed on a circle of length $2\pi\sqrt{\rho}$, hence $f_{\sqrt{\rho}s}(\mathrm{s}) = 1/(2\pi\sqrt{\rho})$ for s over that circle and $\mathfrak{h}(\sqrt{\rho}s) = \log_2(2\pi\sqrt{\rho})$. Therefore, the leading term of (1.332) for large ρ is $\log_2\sqrt{\rho} = \frac{1}{2}\log_2\rho$ as established in Example 1.8. This is only half what a complex Gaussian distribution attains in this regime, the price of ∞-PSK conveying information only in its phase and being (despite its involving both the in-phase and quadrature components) a one-dimensional distribution. Correcting this shortfall requires conveying information also in the magnitude. Moreover, correcting the behavior for $\rho \to \infty$ requires an unbounded number of potential amplitudes, which is the idea behind the signal distribution proposed in this problem.

Let s be uniformly distributed on a disk delimited by magnitudes A_0 and A_1. Then,

$$f_{\sqrt{\rho}s}(\mathrm{s}) = \frac{1}{\pi\rho\,(A_1^2 - A_0^2)} \qquad |\mathrm{s}| \in [A_0, A_1] \tag{1.333}$$

and

$$\mathfrak{h}(\sqrt{\rho}s) = \int_{\sqrt{\rho}A_0}^{\sqrt{\rho}A_1} \int_0^{2\pi} \frac{\log_2\left(\pi\rho\left(A_1^2 - A_0^2\right)\right)}{\pi\rho\left(A_1^2 - A_0^2\right)} |s| \, \mathrm{d}|s| \, \mathrm{d}\phi(s) \tag{1.334}$$

$$= 2 \frac{\log_2\left(\pi\rho\left(A_1^2 - A_0^2\right)\right)}{\rho\left(A_1^2 - A_0^2\right)} \frac{|s|^2}{2}\Big|_{\sqrt{\rho}A_0}^{\sqrt{\rho}A_1} \tag{1.335}$$

$$= \log_2\left(\pi\rho\left(A_1^2 - A_0^2\right)\right). \tag{1.336}$$

Therefore, the leading term of (1.332) for large ρ is $\log_2 \rho$, as with a complex Gaussian signal. Although immaterial to this asymptote, the width of the disk (i.e., the values of A_0 and A_1) do determine how soon this behavior sets in.

The next term in the expansion of $\mathcal{I}(\rho)$, the constant or zero-order term, is sure to be negative so as to reflect the deficit with respect to the high-ρ mutual information for an optimum complex Gaussian signal in (1.53). And, since a circular distribution is closer than a rectangular one to a complex Gaussian, for well-chosen A_0 and A_1 this constant should fall between 0 and $-\log_2(\pi e/6) = -0.51$, the value derived for ∞-QAM in Example 1.9. Obtaining this constant exactly would require an appropriate expansion of $\mathfrak{h}(\sqrt{\rho}s + z)$, in lieu of $\mathfrak{h}(\sqrt{\rho}s)$.

1.19 Repeat Problem 1.18 with s conforming to a one-dimensional discrete constellation featuring M points equidistant along a line forming an angle ϕ with the real axis.

Solution

Since the noise is circularly symmetric, when can apply any arbitrary rotation to the signal, hence without loss of generality we can consider that $\phi = 0$ and we are thus faced with the same signal distribution of Problem 1.4, namely

$$\left\{(2m + 1 - M) \frac{d_{\mathsf{min}}}{2}\right\} \qquad m = 0, \ldots, M-1 \tag{1.337}$$

with

$$d_{\mathsf{min}} = 2\sqrt{\frac{3}{M^2 - 1}}. \tag{1.338}$$

(a) This signal is one-dimensional, not proper complex. Hence, (1.52) need not apply, although as we see next the first-order term does hold.
With s abiding by (1.337), and letting $y = y_{\mathrm{i}} + \mathrm{j} y_{\mathrm{q}}$,

$$f_y(\mathrm{y}) = \frac{1}{\pi M} \sum_{m=1}^{M} e^{-|\mathrm{y} - \sqrt{\rho}\mathrm{s}_m|^2} \tag{1.339}$$

$$= \frac{1}{\pi M} \sum_{m=1}^{M} e^{-(\mathrm{y}_{\mathrm{i}} - \sqrt{\rho}\mathrm{s}_m)^2 - \mathrm{y}_{\mathrm{q}}^2} \tag{1.340}$$

$$= \frac{1}{\pi M} e^{-(\mathrm{y}_\mathrm{i}^2+\mathrm{y}_\mathrm{q}^2)} \sum_{m=1}^{M} e^{2\sqrt{\rho}\mathrm{y}_\mathrm{i}\mathrm{s}_m - \rho \mathrm{s}_m^2} \tag{1.341}$$

$$= \frac{1}{\pi M} e^{-|\mathrm{y}|^2} \sum_{m=1}^{M} \left(1 + 2\sqrt{\rho}\,\mathrm{y}_\mathrm{i}\mathrm{s}_m + \mathcal{O}(\rho)\right) \tag{1.342}$$

$$= \frac{1}{\pi} e^{-|\mathrm{y}|^2} \left(1 + \frac{2\sqrt{\rho}\,\mathrm{y}_\mathrm{i}}{M} \sum_{m=1}^{M} \mathrm{s}_m + \mathcal{O}(\rho)\right) \tag{1.343}$$

$$= \frac{1}{\pi} e^{-|\mathrm{y}|^2} + \mathcal{O}(\rho) \tag{1.344}$$

where the last equality follows from the zero-mean nature of s. The low-ρ differential entropy of y is thus dominated by

$$-\mathbb{E}\left[\log_2\left(\frac{1}{\pi} e^{-|y|^2}\right)\right] = \mathbb{E}[|y|^2] \log_2 e + \log_2 \pi \tag{1.345}$$

$$= \mathbb{E}\left[|\sqrt{\rho}s + z|^2\right] \log_2 e + \log_2 \pi \tag{1.346}$$

$$= (1+\rho) \log_2 e + \log_2 \pi \tag{1.347}$$

$$= \rho \log_2 + \log_2(\pi e), \tag{1.348}$$

meaning the first-order term of $I(s; y)$ is

$$\rho \log_2 e + \log_2(\pi e) - \mathfrak{h}(y|s) = \rho \log_2 e + \log_2(\pi e) - \mathfrak{h}(z) \tag{1.349}$$

$$= \rho \log_2 e + \log_2(\pi e) - \log_2(\pi e) \tag{1.350}$$

$$= \rho \log_2 e. \tag{1.351}$$

(b) Combining (1.68) and (1.338), the leading term for large ρ is

$$\log_2 M - \epsilon \tag{1.352}$$

with

$$\log \epsilon = -\frac{3}{M^2-1}\rho + o(\rho). \tag{1.353}$$

1.20 Let s and z conform to BPSK distributions. Express $\mathcal{I}(\rho) = I(s; \sqrt{\rho}s + z)$ and obtain expansions thereof for small and large ρ. How much is $\mathcal{I}(\rho)$ for $\rho = 5$?

Solution

For $\rho \neq 1$,

$$y = \begin{cases} -1-\sqrt{\rho} & \text{with probability } 1/4 \\ -1+\sqrt{\rho} & \text{with probability } 1/4 \\ 1-\sqrt{\rho} & \text{with probability } 1/4 \\ 1+\sqrt{\rho} & \text{with probability } 1/4 \end{cases} \tag{1.354}$$

such that $\mathcal{H}(y) = 2$. Thus,

$$I(s; \sqrt{\rho}s + z) = 2 - \mathcal{H}(y|s) \tag{1.355}$$
$$= 2 - \mathcal{H}(z) \tag{1.356}$$
$$= 2 - 1 \tag{1.357}$$
$$= 1 \tag{1.358}$$

meaning that $\mathcal{I}(\rho) = 1$ for $\rho \neq 1$. For $\rho = 1$,

$$y = \begin{cases} -2 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ 2 & \text{with probability } 1/4 \end{cases} \tag{1.359}$$

such that $\mathcal{H}(y) = 1.5$. Thus, $\mathcal{I}(\rho) = 0.5$ for $\rho = 1$. Altogether,

$$\mathcal{I}(\rho) = \begin{cases} 0 & \rho = 0 \\ 0.5 & \rho = 1 \\ 1 & \rho > 0,\ \rho \neq 1, \end{cases} \tag{1.360}$$

which, interestingly, coincides for low and high ρ.

1.21 Compute $I(s; \sqrt{\rho}s + z)$ with $s \sim \mathcal{N}_{\mathbb{C}}(0, 1)$ and with z having a BPSK distribution.

Solution

The requested mutual information is given by

$$I(s; \sqrt{\rho}s + z) = \mathfrak{h}(\sqrt{\rho}s + z) - \mathfrak{h}(z) \tag{1.361}$$

where $\mathfrak{h}(\sqrt{\rho}s + z)$ could be expressed as a function of $\mathfrak{h}(\sqrt{\rho}y') = \mathfrak{h}(y') + \log_2 \sqrt{\rho}$ where y' is the output of a channel with BPSK transmit signal, gain $1/\sqrt{\rho}$, and standard complex Gaussian noise. This would allow capitalizing on Example 1.10 to express $\mathfrak{h}(\sqrt{\rho}s + z)$. However, since z is discrete, its mutual information is—as established in Problem 1.7—unboundedly negative and the mutual information is thus infinite.

This should not be a surprising result, as an arbitrarily large number of distinct signals drawn from a complex Gaussian distribution can be distinguished in the face of noise that takes only two possible values. Hence, binary noise does not make a dent in the information that a complex Gaussian signal can pack, which is infinite.

1.22 Compute $I(s; \sqrt{\rho}s + z)$ with both s and z having BPSK distributions.

Solution

For $\rho \neq 1$,

$$y = \begin{cases} -1-\sqrt{\rho} & \text{with probability } 1/4 \\ -1+\sqrt{\rho} & \text{with probability } 1/4 \\ 1-\sqrt{\rho} & \text{with probability } 1/4 \\ 1+\sqrt{\rho} & \text{with probability } 1/4 \end{cases} \tag{1.362}$$

such that $\mathcal{H}(y) = 2$. Thus,

$$I(s; \sqrt{\rho}s + z) = 2 - \mathcal{H}(y|s) \tag{1.363}$$
$$= 2 - \mathcal{H}(z) \tag{1.364}$$
$$= 2 - 1 \tag{1.365}$$
$$= 1 \tag{1.366}$$

meaning that $\mathcal{I}(\rho) = 1$ for $\rho \neq 1$. For $\rho = 1$,

$$y = \begin{cases} -2 & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ 2 & \text{with probability } 1/4 \end{cases} \tag{1.367}$$

such that $\mathcal{H}(y) = 1.5$. Thus, $\mathcal{I}(\rho) = 0.5$ for $\rho = 1$. Altogether,

$$\mathcal{I}(\rho) = \begin{cases} 0 & \rho = 0 \\ 0.5 & \rho = 1 \\ 1 & \rho > 0, \ \rho \neq 1. \end{cases} \tag{1.368}$$

1.23 Verify that, as argued in Example 1.11,

$$\mathcal{I}^{\text{QPSK}}(\rho) = 2\,\mathcal{I}^{\text{BPSK}}\left(\frac{\rho}{2}\right). \tag{1.369}$$

Solution

Visualizing complex quantities as two-dimensional vectors, a BPSK transmission amounts to

$$\begin{bmatrix} y_{\rm i} \\ y_{\rm q} \end{bmatrix} = \sqrt{\rho} \begin{bmatrix} s_{\rm i} \\ 0 \end{bmatrix} + \begin{bmatrix} z_{\rm i} \\ z_{\rm q} \end{bmatrix} \tag{1.370}$$

where $s_{\rm i} = \pm 1$ and the observation of $y_{\rm q}$ is irrelevant (it contains only noise). Then, $\mathcal{I}^{\text{BPSK}}(\rho) = I(s; \sqrt{\rho}s + z) = I(s_{\rm i}; \sqrt{\rho}s_{\rm i} + z_{\rm i})$. In contrast, a QPSK transmission corresponds to

$$\begin{bmatrix} y_{\rm i} \\ y_{\rm q} \end{bmatrix} = \sqrt{\rho} \begin{bmatrix} s_{\rm i} \\ s_{\rm q} \end{bmatrix} + \begin{bmatrix} z_{\rm i} \\ z_{\rm q} \end{bmatrix} \tag{1.371}$$

where $s'_{\rm i} = \pm 1/\sqrt{2}$ and $s'_{\rm q} = \pm 1/\sqrt{2}$. Rewriting the above as

$$\begin{bmatrix} y_{\rm i} \\ y_{\rm q} \end{bmatrix} = \sqrt{\rho/2} \begin{bmatrix} s_{\rm i} \\ s_{\rm q} \end{bmatrix} + \begin{bmatrix} z_{\rm i} \\ z_{\rm q} \end{bmatrix} \tag{1.372}$$

where $s_\mathrm{i} = \pm 1$ and $s_\mathrm{q} = \pm 1$, we obtain two parallel BPSK transmissions with parameter $\rho/2$. As these involve independent signal and noise components, their mutual informations add up, giving

$$\mathcal{I}^{\text{QPSK}}(\rho) = I(s_\mathrm{i}; \sqrt{\rho/2}\, s_\mathrm{i} + z_\mathrm{i}) + I(s_\mathrm{q}; \sqrt{\rho/2}\, s_\mathrm{q} + z_\mathrm{q}) \tag{1.373}$$
$$= 2\, I(s_\mathrm{i}; \sqrt{\rho/2}\, s_\mathrm{i} + z_\mathrm{i}) \tag{1.374}$$
$$= 2\, \mathcal{I}^{\text{BPSK}}(\rho/2). \tag{1.375}$$

The backward version of this result, $\mathcal{I}^{\text{BPSK}}(\rho) = \frac{1}{2}\mathcal{I}^{\text{QPSK}}(2\rho)$, can be explained by arguing that, with BPSK, only half the noise power (the in-phase component) is relevant and the SNR is thus doubled. However, only one of the two signaling dimension of QPSK is used.

1.24 Express the Gaussian mutual information of a square QAM signal as a function of the Gaussian mutual information of another signal whose points are equiprobable and uniformly spaced over the real line.
Note: This relationship substantially simplifies the computation of the Gaussian mutual information of square QAM signals, and it is exploited to perform such computations in this book.

Solution

As (1.2) suggests, a square QAM signal can be obtained as $s = s_\mathrm{i} + \mathrm{j} s_\mathrm{q}$ where s_i and s_q are independent, both drawn from the set

$$\left\{ \sqrt{\tfrac{3}{2\,(M-1)}} \left(2m + 1 - \sqrt{M} \right) \right\} \qquad m = 0, \ldots, \sqrt{M} - 1. \tag{1.376}$$

Applying the logic of Problem 1.23, namely that an M-QAM transmission in complex Gaussian noise amounts to two independent $\sqrt{M}$-PAM transmissions, each with half the SNR, we have that

$$\mathcal{I}^{M\text{-QAM}}(\rho) = 2\, \mathcal{I}^{\sqrt{M}\text{-PAM}}\left(\frac{\rho}{2}\right) \tag{1.377}$$

with the PAM constellation being the one considered in Problems 1.4 and 1.19. An M-PAM constellation is defined by

$$\mathrm{s}_m = \left\{ \sqrt{\tfrac{3}{M^2-1}} \left(2m + 1 - M \right) \right\} \qquad m = 0, \ldots, M - 1 \tag{1.378}$$

and its Gaussian mutual information is given by

$$\mathcal{I}^{M\text{-PAM}}(\rho) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_y(\mathrm{y}) \log_2 f_y(\mathrm{y})\, \mathrm{dy} - \log_2(\pi e) \tag{1.379}$$

with

$$f_y(\mathrm{y}) = \frac{1}{M\pi} \sum_{m=0}^{M-1} e^{-|\mathrm{y} - \sqrt{\rho}\mathrm{s}_m|^2}. \tag{1.380}$$

Exploiting the fact that PAM points have no quadrature component, meaning that only the in-phase noise is relevant, (1.379) can be reduced to a single integral on the real line. To do so, we first let $y = y_{\rm i} + {\rm j} y_{\rm q}$ and rewrite (1.380) as

$$f_y(\mathrm{y}) = \frac{e^{-\mathrm{y}_{\rm q}^2}}{M\pi} \sum_{m=0}^{M-1} e^{-(\mathrm{y}_{\rm i} - \sqrt{\rho}\mathrm{s}_m)^2} \tag{1.381}$$

from which

$$-\log_2 f_y(\mathrm{y}) = \log_2(\pi) + |y_{\rm q}|^2 \log_2 e - \log_2 \sum_{m=0}^{M-1} \frac{e^{-(\mathrm{y}_{\rm i} - \sqrt{\rho}\mathrm{s}_m)^2}}{M} \tag{1.382}$$

and, since $y_{\rm q} = z_{\rm q}$ with $\mathbb{E}[|z_{\rm q}|^2] = 1/2$,

$$\mathcal{I}^{M\text{-PAM}}(\rho) = -\mathbb{E}[\log_2 f_y(y)] - \log_2(\pi e) \tag{1.383}$$

$$= -\int_{-\infty}^{\infty} \frac{e^{-\mathrm{y}_{\rm q}^2}}{\sqrt{\pi}} \mathrm{dy}_{\rm q} \int_{-\infty}^{\infty} \sum_{m=0}^{M-1} \frac{e^{-(\mathrm{y}_{\rm i} - \sqrt{\rho}\mathrm{s}_m)^2}}{M\sqrt{\pi}} \log_2 \sum_{m=0}^{M-1} \frac{e^{-(\mathrm{y}_{\rm i} - \sqrt{\rho}\mathrm{s}_m)^2}}{M} \mathrm{dy}_{\rm i} - \frac{1}{2}\log_2 e \tag{1.384}$$

$$= \frac{-1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \sum_{m=0}^{M-1} \frac{e^{-(\mathrm{y}_{\rm i} - \sqrt{\rho}\mathrm{s}_m)^2}}{M} \log_2 \sum_{m=0}^{M-1} \frac{e^{-(\mathrm{y}_{\rm i} - \sqrt{\rho}\mathrm{s}_m)^2}}{M} \mathrm{dy}_{\rm i} - \frac{1}{2}\log_2 e. \tag{1.385}$$

Therefore,

$$\mathcal{I}^{M\text{-QAM}}(\rho) = \frac{-2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \sum_{m=0}^{M-1} \frac{e^{-(\xi - \sqrt{\rho/2}\mathrm{s}_m)^2}}{M} \log_2 \sum_{m=0}^{M-1} \frac{e^{-(\xi - \sqrt{\rho/2}\mathrm{s}_m)^2}}{M} \mathrm{d}\xi - \log_2 e. \tag{1.386}$$

1.25 Let $y = \sqrt{\rho} s + z$. If z were not independent of s, would that increase or decrease $I(s; y)$ relative to the usual situation where they are independent? Can you draw any communication-theoretic lesson from this?

Solution

Recalling that $I(s; y)$ can be interpreted as the reduction in uncertainty about the value of y that occurs when s becomes known, any dependence between z and s would increase $I(s; y)$ because knowing s would then reveal even more about y. (In the limit, if $z = s$, knowing s would determine y exactly and the reduction in uncertainty would be complete; the mutual information would then be infinite.)

From the above consideration it follows that, everything else being equal, the noise is the most deleterious when it is independent of the signal.

1.26 Let $\boldsymbol{s} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{I})$ and $\boldsymbol{z} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{I})$ while

$$\boldsymbol{A} = \begin{bmatrix} 0.7 & 1+0.5\,\mathrm{j} & 1.2\,\mathrm{j} \\ 0.2+\mathrm{j} & -2.1 & 0 \end{bmatrix}. \tag{1.387}$$

(a) Plot the exact $I(\boldsymbol{s}; \sqrt{\rho}\boldsymbol{A}\boldsymbol{s} + \boldsymbol{z})$ against its low-ρ expansion for $\rho \in [0, 1]$. Up to which value of ρ is the difference below 10%?

(b) Plot the exact $I(\boldsymbol{s}; \sqrt{\rho}\boldsymbol{A}\boldsymbol{s} + \boldsymbol{z})$ against its high-ρ expansion for $\rho \in [10, 100]$. Beyond which value of ρ is the difference below 10%?

Solution

(a) From Example 1.13, the exact mutual information equals

$$\mathcal{I}(\rho) = \log_2 \det(\boldsymbol{I} + \rho \boldsymbol{A}\boldsymbol{A}^*) \tag{1.388}$$

whereas the low-ρ expansion is

$$\begin{aligned} \mathcal{I}(\rho) &= \left[\mathrm{tr}(\boldsymbol{A}\boldsymbol{A}^*)\,\rho - \frac{1}{2}\,\mathrm{tr}(\boldsymbol{A}\boldsymbol{A}^*\boldsymbol{A}\boldsymbol{A}^*)\,\rho^2 \right] \log_2 e + o\left(\rho^2\right) && (1.389) \\ &= 12.45\,\rho - 38.7\,\rho^2 + o\left(\rho^2\right). && (1.390) \end{aligned}$$

The two are depicted in Fig. 1.8, respectively in solid and in dashed, for $\rho \in [0, 1]$. The more familiar form of this plot, with ρ in dB, is provided in Fig. 1.9. The difference is below 10% of the exact mutual information up to $\rho = 0.04$, meaning -10.2 dB. Although this would seem to indicate that low-SNR asymptotics have restricted validity, in Chapter 4 we see how their range of validy spans a much more relevant range once the performance is expressed as a function of the per-bit SNR, rather than the per-symbol SNR ρ.

(b) Again from Example 1.13, since $\boldsymbol{A}\boldsymbol{A}^*$ is nonsingular, the high-ρ expansion is

$$\mathcal{I}(\rho) = 2\,\log_2 \rho + \log_2 \det(\boldsymbol{A}\boldsymbol{A}^*) + \mathcal{O}\left(\frac{1}{\rho}\right), \tag{1.391}$$

which Fig. 1.10 depicts in dashed, alongside the exact mutual information. The more familiar form of this plot, with ρ in dB, is provided in Fig. 1.11. The agreement is excellent, with the difference being less than 10% of the exact mutual information down to $\rho = 1.82$, which is only 2.6 dB.

1.27 Let $\boldsymbol{s}$ have two independent unit-variance entries and let $\boldsymbol{z} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{I})$ while $\boldsymbol{A} = [0.7 \quad 1+0.5\,\mathrm{j}]$. On a common chart, plot $\mathcal{I}(\rho) = I(s; \sqrt{\rho}\boldsymbol{A}\boldsymbol{s} + \boldsymbol{z})$ for $\rho \in [0, 10]$ under the following distributions for the entries of $\boldsymbol{s}$:

(a) Real Gaussian.

(b) Complex Gaussian.

(c) BPSK.

(d) QPSK.

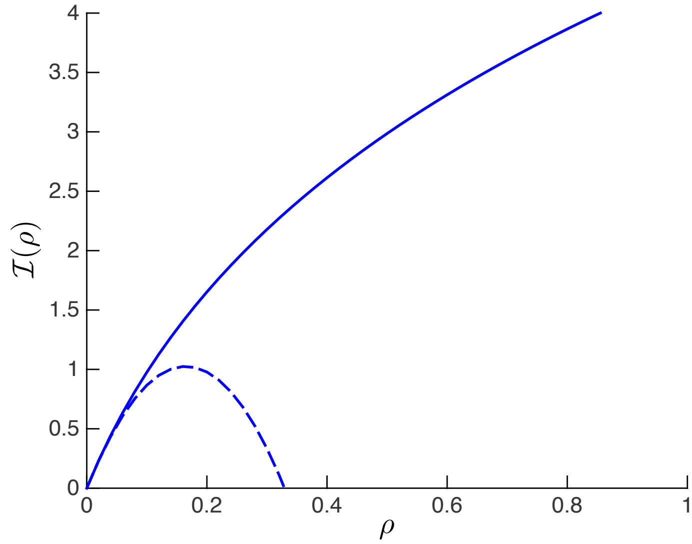

Fig. 1.8 $\mathcal{I}(\rho)$ for the channel in Problem 1.26, in solid, versus its low-ρ second-order expansion, in dashed.

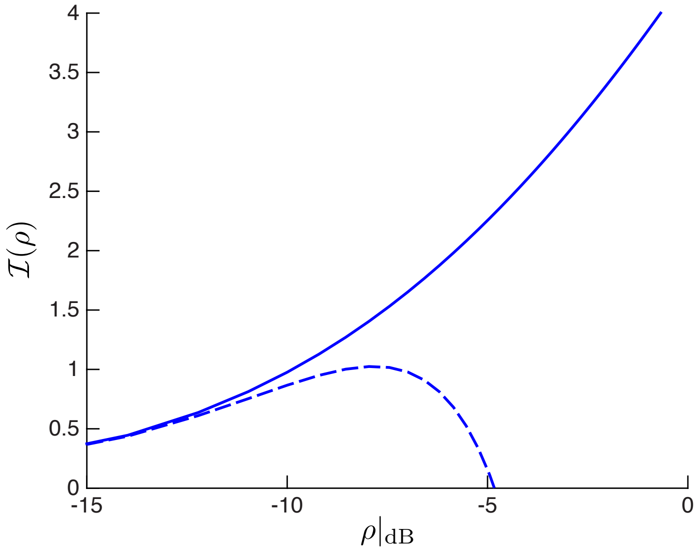

Fig. 1.9 $\mathcal{I}(\rho|_{\text{dB}})$ for the channel in Problem 1.26, in solid, versus its low-ρ second-order expansion, in dashed.

Solution

(a) Applying the reasoning of Problems 1.23 and 1.24, we can conclude that the Gaussian mutual information achieved by a real Gaussian signal is half the value

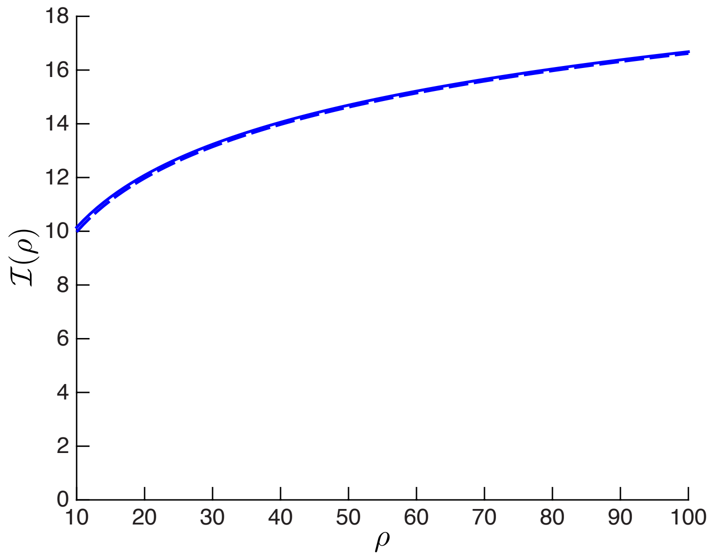

Fig. 1.10 $\mathcal{I}(\rho)$ for the channel in Problem 1.26, in solid, versus its high-ρ second-order expansion, in dashed.

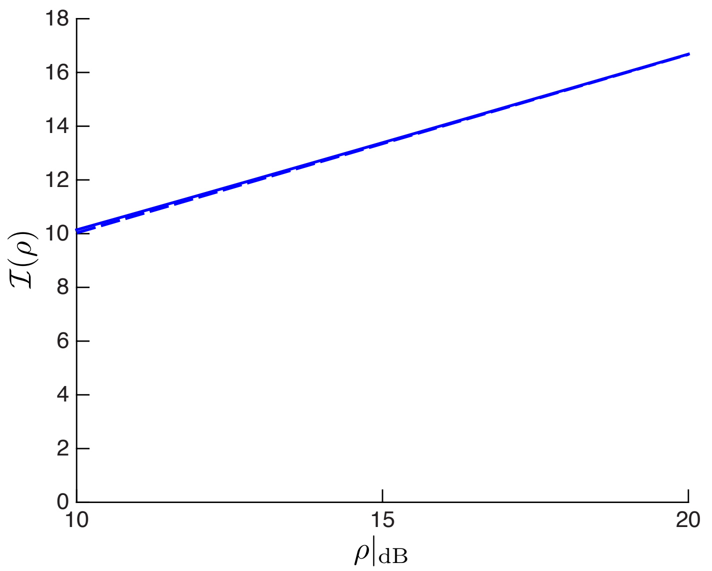

Fig. 1.11 $\mathcal{I}(\rho|_{\mathrm{dB}})$ for the channel in Problem 1.26, in solid, versus its high-ρ second-order expansion, in dashed.

achieved by its complex Gaussian counterpart, but evaluated at twice the SNR.

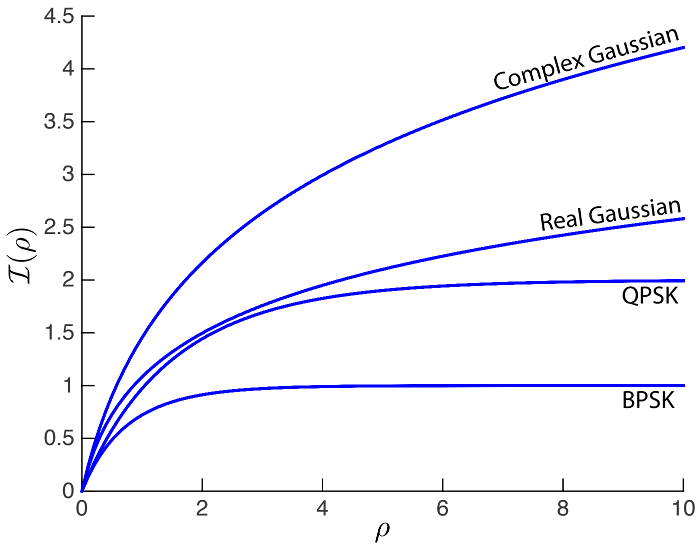

Fig. 1.12 $\mathcal{I}(\rho)$ for the channel in Problem 1.27 with real Gaussian, complex Gaussian, BPSK, and QPSK signaling.

Hence, for $\boldsymbol{s} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I})$,

$$\mathcal{I}(\rho) = \frac{1}{2} \log_2(1 + 2\,\rho\, \boldsymbol{A}\boldsymbol{A}^*) \tag{1.392}$$

$$= \frac{1}{2} \log_2(1 + 3.48\,\rho). \tag{1.393}$$

(b) For $\boldsymbol{s} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{I})$,

$$\mathcal{I}(\rho) = \log_2(1 + 1.74\,\rho). \tag{1.394}$$

(c) For BPSK, $\mathcal{I}^{\text{BPSK}}(\rho)$ is given in Example 1.10. Its evaluation requires a single integration, which can be effected numerically with either MATLAB® or Mathematica®.

(d) For QPSK, $\mathcal{I}^{\text{QPSK}}(\rho) = 2\,\mathcal{I}^{\text{BPSK}}(\rho/2)$.

The Gaussian mutual information functions for the various signals are depicted in Fig. 1.12, for $\rho \in [0, 10]$. The more familiar form for this plot, with ρ in dB, is presented in Fig. 1.13 for $\rho|_{\text{dB}} \in [-5, 10]$ dB.

1.28 Compute and plot, as function of $\rho \in [-5, 25]$ dB, the Gaussian mutual information function for the following constellations:

(a) 8-PSK.

(b) 16-QAM.