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Solutions for Chapter 1

Problem 1.1

In terms of the Cartesian components we have $\mathbf{r} = (x, y, z)$ and $\mathbf{p} = (p_x, p_y, p_z)$. Using $\mathbf{p} = -i\nabla$ and taking any differentiable function $\psi(\mathbf{r})$ of position it follows that we have

$$
\begin{aligned}
[\exp(-i\mathbf{k} \cdot \mathbf{r}), \mathbf{p}]\psi(\mathbf{r}) &= -i\exp(-i\mathbf{k} \cdot \mathbf{r})\nabla\psi(\mathbf{r}) + i\nabla\{\exp(-i\mathbf{k} \cdot \mathbf{r})\psi(\mathbf{r})\} \\
&= -i\exp(-i\mathbf{k} \cdot \mathbf{r})\nabla\psi(\mathbf{r}) + i\exp(-i\mathbf{k} \cdot \mathbf{r})\nabla\psi(\mathbf{r}) \\
&\quad +i(-i\mathbf{k})\exp(-i\mathbf{k} \cdot \mathbf{r})\psi(\mathbf{r}) \\
&= \mathbf{k}\exp(-i\mathbf{k} \cdot \mathbf{r})\psi(\mathbf{r}).
\end{aligned}
$$

Since this holds for any $\psi(\mathbf{r})$ we obtain

$$
[\exp(-i\mathbf{k} \cdot \mathbf{r}), \mathbf{p}] = \mathbf{k}\exp(-i\mathbf{k} \cdot \mathbf{r}).
$$

Then we consider $[\exp(-i\mathbf{k} \cdot \mathbf{r}), \mathbf{p}^2]$, which gives

$$
\begin{aligned}
[\exp(-i\mathbf{k} \cdot \mathbf{r}), \mathbf{p}^2] &= \mathbf{p} \cdot [\exp(-i\mathbf{k} \cdot \mathbf{r}), \mathbf{p}] + [\exp(-i\mathbf{k} \cdot \mathbf{r}), \mathbf{p}] \cdot \mathbf{p} \\
&= \mathbf{k} \cdot \mathbf{p}\exp(-i\mathbf{k} \cdot \mathbf{r}) + \exp(-i\mathbf{k} \cdot \mathbf{r})\mathbf{k} \cdot \mathbf{p},
\end{aligned}
$$

where we made use of the result from the first part of the question in order to obtain the above line. The first term can be rewritten as

$$
\mathbf{k} \cdot \mathbf{p}\exp(-i\mathbf{k} \cdot \mathbf{r}) = \exp(-i\mathbf{k} \cdot \mathbf{r})\mathbf{k} \cdot \mathbf{p} - \exp(-i\mathbf{k} \cdot \mathbf{r})\mathbf{k} \cdot \mathbf{k},
$$

which leads to the final result that

$$
[\exp(-i\mathbf{k} \cdot \mathbf{r}), \mathbf{p}^2] = \exp(-i\mathbf{k} \cdot \mathbf{r})(2\mathbf{k} \cdot \mathbf{p} - k^2).
$$

Problem 1.2

We start with the commutator between $\mathbf{A}_n(\mathbf{r}, t)$ and $\mathbf{E}_m(\mathbf{r}', t)$ by using the expressions in equations (1.15) and (1.16). The result is

$$
\begin{aligned}
[\mathbf{A}_n(\mathbf{r}, t), \mathbf{E}_m(\mathbf{r}', t)] = &\sum_{l,l'} \sum_{\lambda,\lambda'=1,2} \frac{i}{2\varepsilon_0 V} \sqrt{\frac{\omega_{l'}}{\omega_l}} (\hat{e}_{l,\lambda})_n (\hat{e}_{l',\lambda'})_m \\
&\times \Big[\Big(a_l e^{i(\mathbf{k}_l \cdot \mathbf{r} - \omega_l t)} + a_l^\dagger e^{-i(\mathbf{k}_l \cdot \mathbf{r} - \omega_l t)} \Big), \Big(a_{l'} e^{i(\mathbf{k}_{l'} \cdot \mathbf{r}' - \omega_{l'} t)} - a_{l'}^\dagger e^{-i(\mathbf{k}_{l'} \cdot \mathbf{r}' - \omega_{l'} t)} \Big) \Big] \\
= &\sum_{l,l'} \sum_{\lambda,\lambda'=1,2} \frac{i}{2\varepsilon_0 V} \sqrt{\frac{\omega_{l'}}{\omega_l}} (\hat{e}_{l,\lambda})_n (\hat{e}_{l',\lambda'})_m \Big\{ \Big[a_l e^{i(\mathbf{k}_l \cdot \mathbf{r} - \omega_l t)}, -a_{l'}^\dagger e^{-i(\mathbf{k}_{l'} \cdot \mathbf{r}' - \omega_{l'} t)} \Big] \\
&+ \Big[a_l^\dagger e^{-i(\mathbf{k}_l \cdot \mathbf{r} - \omega_l t)}, a_{l'} e^{i(\mathbf{k}_{l'} \cdot \mathbf{r}' - \omega_{l'} t)} \Big] \Big\} \\
= &\sum_l \sum_{\lambda,\lambda'=1,2} \frac{-i}{2\varepsilon_0 V} (\hat{e}_{l,\lambda})_n (\hat{e}_{l,\lambda'})_m \Big(\exp(i\mathbf{k}_l \cdot \mathbf{R}) + \exp(-i\mathbf{k}_l \cdot \mathbf{R}) \Big),
\end{aligned}
$$

where we have denoted $\mathbf{R} = \mathbf{r} - \mathbf{r}'$. Now, since the quantities $\{\hat{e}_{l,1}, \hat{e}_{l,2}, \hat{\mathbf{k}}_l\}$ form an orthogonal basic set in the 3D space, it follows that we can write $|\hat{e}_{l,1}\rangle\langle\hat{e}_{l,1}| + |\hat{e}_{l,2}\rangle\langle\hat{e}_{l,2}| + |\hat{\mathbf{k}}_l\rangle\langle\hat{\mathbf{k}}_l| = 1$, implying

$$
\sum_{\lambda=1,2} (\hat{e}_{l,\lambda})_n (\hat{e}_{l,\lambda})_m = \delta_{n,m} - \frac{(\mathbf{k}_l)_n (\mathbf{k}_l)_m}{k_l^2} \equiv \delta_{n,m} - (\hat{\mathbf{k}}_l)_n (\hat{\mathbf{k}}_l)_m.
$$

After substituting the above result into the previous commutator expression we have

$$
[\mathbf{A}_n(\mathbf{r}, t), \mathbf{E}_m(\mathbf{r}', t)] = \sum_l \frac{-i}{\varepsilon_0 V} \Big(\delta_{n,m} - (\hat{\mathbf{k}}_l)_n (\hat{\mathbf{k}}_l)_m \Big) \exp(i\mathbf{k}_l \cdot \mathbf{R}).
$$

We note that the summation over l now covers both positive and negative values, since $\hat{\mathbf{k}}_{-l} = -\hat{\mathbf{k}}_l$. If we now let V become macroscopically large $(V \to \infty)$, this summation can be replaced by an integral over the 3D wave vector using

$$
\frac{1}{V} \sum_l \cdots \to \Big(\frac{1}{2\pi} \Big)^3 \int \cdots d^3 k,
$$

and we arrive at the final result as

$$
[\mathbf{A}_n(\mathbf{r}, t), \mathbf{E}_m(\mathbf{r}', t)] = \frac{-i}{\varepsilon_0} \delta_{n,m}^T (\mathbf{r} - \mathbf{r}').
$$

Here we have introduced the transverse delta function δ^T, which is defined generally as

$$
\delta_{n,m}^T (\mathbf{r} - \mathbf{r}') = \frac{1}{(2\pi)^3} \int d^3 k \Big(\delta_{n,m} - (\hat{\mathbf{k}})_n (\hat{\mathbf{k}})_m \Big) \exp(i\mathbf{k} \cdot \mathbf{R}).
$$

By following the same procedure for the other commutators and using the appropriate terms in equations (1.15) and (1.16), we can find that

$$[\mathbf{A}_n(\mathbf{r}, t), \mathbf{A}_m(\mathbf{r}', t)] = 0, \qquad [\mathbf{H}_n(\mathbf{r}, t), \mathbf{H}_m(\mathbf{r}', t)] = 0,$$

$$[\mathbf{E}_n(\mathbf{r}, t), \mathbf{E}_m(\mathbf{r}', t)] = 0, \qquad [\mathbf{H}_n(\mathbf{r}, t), \mathbf{A}_m(\mathbf{r}', t)] = 0.$$

Problem 1.3

We start by considering the commutation relation $[a, (a^\dagger)^n]$, where n is a positive integer. This leads to

$$
\begin{aligned}
[a, (a^\dagger)^n] &= [a, a^\dagger (a^\dagger)^{(n-1)}] = [a, a^\dagger]\, a^{\dagger(n-1)} + a^\dagger \left[a, a^{\dagger(n-1)}\right] \\
&= (a^\dagger)^{(n-1)} + a^\dagger [a, a^\dagger (a^\dagger)^{(n-2)}] \\
&= (a^\dagger)^{(n-1)} + (a^\dagger)^{(n-1)} + (a^\dagger)^2 [a, (a^\dagger)^{(n-3)}] \\
&= (a^\dagger)^{(n-1)} + (a^\dagger)^{(n-1)} + \cdots + (a^\dagger)^{(n-1)} [a, a^\dagger] \\
&= n(a^\dagger)^{(n-1)} = \frac{\partial (a^\dagger)^n}{\partial a^\dagger}.
\end{aligned}
$$

In a similar way it is found that

$$[a^\dagger, a^n] = -na^{(n-1)} = -\frac{\partial a^n}{\partial a}.$$

It follows, therefore, that for any function $f(a, a^\dagger)$ expressible as a Taylor series expansion in positive integer powers of a and $a^\dagger$ we obtain

$$\left[a, f(a, a^\dagger)\right] = \frac{\partial f(a, a^\dagger)}{\partial a^\dagger},$$

$$\left[a^\dagger, f(a, a^\dagger)\right] = -\frac{\partial f(a, a^\dagger)}{\partial a},$$

as required.

For the last part of this problem we may start by considering

$$
\begin{aligned}
\{\exp{(\lambda A)}\, B \exp{(-\lambda A)}\}^n & \\
&= \exp{(\lambda A)}\, B \exp{(-\lambda A)} \exp{(\lambda A)}\, B \exp{(-\lambda A)}\, \cdots \\
&= \exp{(\lambda A)}\, B^n \exp{(-\lambda A)}
\end{aligned}
$$

for any operators A and B and any constant λ. It then follows, for a more general case when we have any function $f(B)$ which may be expanded as a power series in B, that we have

$$f\Big(\exp{(\lambda A)}\, B \exp{(-\lambda A)} \Big) = \exp{(\lambda A)}\, f(B) \exp{(-\lambda A)}.$$

Therefore, taking $\lambda A \equiv -\alpha\, a^\dagger a$ and $f(B) \equiv f(a^\dagger, a)$ in the above expression, we may conclude that

$$\exp(-\alpha\, a^\dagger a) f(a, a^\dagger) \exp(\alpha a^\dagger a)$$
$$= f\Big(\exp(-\alpha a^\dagger a) a \exp(\alpha a^\dagger a), \exp(-\alpha a^\dagger a) a^\dagger \exp(\alpha a^\dagger a) \Big).$$

The final step is to show that the functional dependences simplify because $\exp(-\alpha a^\dagger a) a \exp(\alpha a^\dagger a) = a e^\alpha$ and $\exp(-\alpha a^\dagger a) a^\dagger \exp(\alpha a^\dagger a) = a^\dagger e^{-\alpha}$. These relationships can easily be proved from first principles by expanding the exponential operator in powers of the exponents and then simplifying using the boson commutation properties. A more elegant alternative is to use the Baker-Campbell-Hausdorff identity quoted in section 8.2 of this book.

Then, using the above results we obtain the required result that

$$\exp(-\alpha a^\dagger a) f(a, a^\dagger) \exp(\alpha a^\dagger a) = f(a e^\alpha, a^\dagger e^{-\alpha}).$$

Problem 1.4

(a) We start by defining a quantity $f(\lambda)$, where λ is a real variable, by

$$f(\lambda) = \exp(\lambda A)\, B \exp(-\lambda A).$$

We now differentiate with respect to λ, obtaining

$$\frac{d}{d\lambda} f(\lambda) = \exp(\lambda A)\, (AB - BA) \exp(-\lambda A)$$
$$= \exp(\lambda A)\, [A, B] \exp(-\lambda A).$$

Since $[A, B] = c$ (a scalar constant) we have

$$\frac{d}{d\lambda} f(\lambda) = c,$$

while $d^2 f/d\lambda^2$ and all higher derivatives are zero. From the Taylor series expansion of $f(\lambda)$ we have

$$f(\lambda) = f(0) + \lambda \frac{d}{d\lambda} f(0) + \frac{\lambda^2}{2!} \frac{d^2}{d\lambda^2} f(0) + \cdots,$$

which simplifies to become

$$\exp(\lambda A)\, B \exp(-\lambda A) = B + \lambda\, [A, B].$$

(b) In this case we may define the shorthand

$$g(\lambda) = \exp(\lambda A) \exp(\lambda B).$$

Therefore, on differentiating with respect to λ, we find

$$
\begin{aligned}
\frac{d}{d\lambda} g(\lambda) &= A \exp\left(\lambda A\right) \exp\left(\lambda B\right) + \exp\left(\lambda A\right) B \exp\left(\lambda B\right) \\
&= A \exp\left(\lambda A\right) \exp\left(\lambda B\right) + (B + \lambda c) \exp\left(\lambda A\right) \exp\left(\lambda B\right) \\
&= (A + B + \lambda c) \, g(\lambda) \\
&= (A + B + \lambda\left[A, B\right]) \, g(\lambda),
\end{aligned}
$$

where in the last step we have used the result from part (a).

(c) Using the above definition of $g(\lambda)$ we can re-derive the result in part (b) for $dg(\lambda)/d\lambda$ in a different way as

$$
\begin{aligned}
\frac{d}{d\lambda} g(\lambda) &= \exp\left(\lambda A\right) A \exp\left(\lambda B\right) + \exp\left(\lambda A\right) \exp\left(\lambda B\right) B \\
&= \exp\left(\lambda A\right) \exp\left(\lambda B\right) \left[\exp\left(-\lambda B\right) A \exp\left(\lambda B\right) + B\right] \\
&= g(\lambda) \left(A + B + \lambda\left[A, B\right]\right).
\end{aligned}
$$

By comparison of this result with the result in part (b) we establish that $g(\lambda)$ commutes with $A + B + \lambda[A, B]$, so we can integrate either differential equation to obtain

$$
g(\lambda) = \exp\left((A + B)\lambda + \frac{\lambda^2}{2}\left[A, B\right]\right) = \exp\lambda(A + B)\exp\frac{\lambda^2}{2}\left[A, B\right],
$$

which is the required result.

Finally, by substituting $\lambda = 1$, $A = \alpha a^\dagger$ and $B = -\alpha^* a$ (which leads to the commutator $[A, B] = -|\alpha|^2[a^\dagger, a] = |\alpha|^2$) into the result of part (c), we obtain initially

$$
e^{\alpha a^\dagger} e^{-\alpha^* a} = \exp(\alpha a^\dagger - \alpha^* a + |\alpha|^2/2).
$$

This can be rearranged as

$$
D(\alpha) \equiv \exp(\alpha a^\dagger - \alpha^* a) = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} e^{-\alpha^* a},
$$

as required.

Problem 1.5

From equation (1.39) the coherent state $|\alpha\rangle$ can be written as

$$
|\alpha\rangle = e^{-\alpha\alpha^*/2} e^{\alpha a^\dagger} |0\rangle \,,
$$

and therefore we have

$$|\alpha\rangle \langle\alpha| = e^{-\alpha\alpha^*} e^{\alpha a^\dagger} |0\rangle \langle0| e^{\alpha^* a}.$$

Now we may obtain the partial derivative with respect to α as

$$
\begin{aligned}
\frac{\partial}{\partial\alpha} |\alpha\rangle \langle\alpha| &= -\alpha^* e^{-\alpha\alpha^*} e^{\alpha a^\dagger} |0\rangle \langle0| e^{\alpha^* a} + e^{-\alpha\alpha^*} a^\dagger e^{\alpha a^\dagger} |0\rangle \langle0| e^{\alpha^* a} \\
&= \left(-\alpha^* + a^\dagger\right) e^{-\alpha\alpha^*} e^{\alpha a^\dagger} |0\rangle \langle0| e^{\alpha^* a} \\
&= \left(-\alpha^* + a^\dagger\right) |\alpha\rangle \langle\alpha|.
\end{aligned}
$$

Combining these last two expressions, it follows that

$$\left(\frac{\partial}{\partial\alpha} + \alpha^*\right) |\alpha\rangle \langle\alpha| = a^\dagger |\alpha\rangle \langle\alpha|.$$

Similarly, on partially differentiating the expression for $|\alpha\rangle\langle\alpha|$ with respect to α^*, we obtain

$$
\begin{aligned}
\frac{\partial}{\partial\alpha^*} |\alpha\rangle \langle\alpha| &= -\alpha e^{-\alpha\alpha^*} e^{\alpha a^\dagger} |0\rangle \langle0| e^{\alpha^* a} + e^{-\alpha\alpha^*} e^{\alpha a^\dagger} |0\rangle \langle0| a e^{\alpha^* a} \\
&= -\alpha e^{-\alpha\alpha^*} e^{\alpha a^\dagger} |0\rangle \langle0| e^{\alpha^* a} + e^{-\alpha\alpha^*} e^{\alpha a^\dagger} |0\rangle \langle0| e^{\alpha^* a} a \\
&= |\alpha\rangle \langle\alpha| \left(-\alpha + a\right).
\end{aligned}
$$

In the above we have used the property that the commutator $[a, e^{\alpha^* a}] = 0$, and therefore we have

$$\left(\frac{\partial}{\partial\alpha^*} + \alpha\right) |\alpha\rangle \langle\alpha| = |\alpha\rangle \langle\alpha| a.$$

Problem 1.6

We take the volume to be a cube with sides of length L (and volume $V = L^3$), where the sides are parallel to the x, y, z axes. Because of the periodic boundary conditions applied over a length L we need to have (e.g., in the x direction)

$$n_x\left(\frac{2\pi}{k_x}\right) = L, \quad \text{so} \quad k_x = \frac{2\pi}{L} n_x,$$

where n_x is an integer. There are similar results for the wavenumbers in the y and z directions. The required integral is

$$
\begin{aligned}
\frac{1}{V} \int d^3r\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} &= \frac{1}{L^3} \int\int\int dx\, dy\, dz\, e^{i(k_x-k_x')x} e^{i(k_y-k_y')y} e^{i(k_z-k_z')z} \\
&= \frac{1}{L} \int_0^L dx\, e^{i(k_x-k_x')x} \frac{1}{L} \int_0^L dy\, e^{i(k_y-k_y')y} \frac{1}{L} \int_0^L dz\, e^{i(k_z-k_z')z}.
\end{aligned}
$$