PROBLEM SET 2.1

Problem 1

(a)

$$\begin{aligned}|\mathbf{A}| &= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 1\begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \\ &= 1(-1) - 2(-2) + 3(-1) = 0 \quad \text{Singular} \blacktriangleleft\end{aligned}$$

(b)

$$\begin{aligned}|\mathbf{A}| &= \begin{vmatrix} 2.11 & -0.80 & 1.72 \\ -1.84 & 3.03 & 1.29 \\ -1.57 & 5.25 & 4.30 \end{vmatrix} \\ &= 2.11\begin{vmatrix} 3.03 & 1.29 \\ 5.25 & 4.30 \end{vmatrix} + 0.80\begin{vmatrix} -1.84 & 1.29 \\ -1.57 & 4.30 \end{vmatrix} + 1.72\begin{vmatrix} -1.84 & 3.03 \\ -1.57 & 5.25 \end{vmatrix} \\ &= 2.11(6.2565) + 0.80(-5.8867) + 1.72(-4.9029) \\ &= 0.058\,867 \quad \text{Ill conditioned} \blacktriangleleft\end{aligned}$$

(c)

$$\begin{aligned}|\mathbf{A}| &= \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + 1\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} \\ &= 2(3) + 1(-2) = 4 \quad \text{Well-conditioned} \blacktriangleleft\end{aligned}$$

(d)

$$\begin{aligned}|\mathbf{A}| &= \begin{vmatrix} 4 & 3 & -1 \\ 7 & -2 & 3 \\ 5 & -18 & 13 \end{vmatrix} = 4\begin{vmatrix} -2 & 3 \\ -18 & 13 \end{vmatrix} - 3\begin{vmatrix} 7 & 3 \\ 5 & 13 \end{vmatrix} - 1\begin{vmatrix} 7 & -2 \\ 5 & -18 \end{vmatrix} \\ &= 4(28) - 3(76) - 1(-116) = 0 \quad \text{Singular} \blacktriangleleft\end{aligned}$$

Problem 2

(a)

$$\mathbf{A} = \mathbf{LU} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 5/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 21 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 5 & 25 \\ 1 & 7 & 39 \end{bmatrix} \quad \blacktriangleleft$$

$$|\mathbf{A}| = |\mathbf{L}|\,|\mathbf{U}| = (1 \times 1 \times 1)(1 \times 3 \times 0) = 0 \quad \blacktriangleleft$$

(b)

$$\mathbf{A} = \mathbf{LU} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 2 & -4 \\ 2 & -4 & 11 \end{bmatrix} \quad \blacktriangleleft$$

$$|\mathbf{A}| = |\mathbf{L}|\,|\mathbf{U}| = (2 \times 1 \times 1)(2 \times 1 \times 1) = 4 \quad \blacktriangleleft$$

Problem 3

First solve $\mathbf{Ly} = \mathbf{b}$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 1/2 & 11/13 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\begin{aligned} y_1 &= 1 \\ \frac{3}{2}(1) + y_2 &= -1 \qquad y_2 = -\frac{5}{2} \\ \frac{1}{2}(1) + \frac{11}{13}\left(\frac{-5}{2}\right) + y_3 &= 2 \qquad y_3 = \frac{47}{13} \end{aligned}$$

Then solve $\mathbf{Ux} = \mathbf{y}$:

$$\begin{bmatrix} 2 & -3 & -1 \\ 0 & 13/2 & -7/2 \\ 0 & 0 & 32/13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -5/2 \\ 47/13 \end{bmatrix}$$

$$\begin{aligned} \frac{32}{13}x_3 &= \frac{47}{13} \qquad x_3 = \frac{47}{32} \quad \blacktriangleleft \\ \frac{13}{2}x_2 - \frac{7}{2}\left(\frac{47}{32}\right) &= -\frac{5}{2} \qquad x_2 = \frac{13}{32} \quad \blacktriangleleft \\ 2x_1 - 3\left(\frac{13}{32}\right) - \frac{47}{32} &= 1 \qquad x_1 = \frac{59}{32} \quad \blacktriangleleft \end{aligned}$$

Problem 4

The augmented coefficient matrix is

$$[\mathbf{A}|\mathbf{b}] = \begin{bmatrix} 2 & -3 & -1 & 3 \\ 3 & 2 & -5 & -9 \\ 2 & 4 & -1 & -5 \end{bmatrix}$$

Elimination phase:

$$\begin{aligned} \text{row } 2 &\leftarrow \text{row } 2 - \frac{3}{2} \times \text{row } 1 \\ \text{row. } 3 &\leftarrow \text{row } 3 - \text{row } 1 \end{aligned}$$

$$\begin{bmatrix} 2 & -3 & -1 & 3 \\ 3-(3/2)(2) & 2-(3/2)(-3) & -5-(3/2)(-1) & -9-(3/2)(3) \\ 2-2 & 4-(-3) & -1-(-1) & -5-3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -3 & -1 & 3 \\ 0 & 13/2 & -7/2 & -27/2 \\ 0 & 7 & 0 & -8 \end{bmatrix}$$

$$\text{row } 3 \leftarrow \text{row } 3 - \frac{14}{13} \times \text{row } 2$$

$$\begin{bmatrix} 2 & -3 & -1 & 3 \\ 0 & 13/2 & -7/2 & -27/2 \\ 0 & 7-(14/13)(13/2) & 0-(14/13)(-7/2) & -8-(14/13)(-27/2) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -3 & -1 & 3 \\ 0 & 13/2 & -7/2 & -27/2 \\ 0 & 0 & 49/13 & 85/13 \end{bmatrix}$$

Solution by back substitution:

$$\begin{aligned} \frac{49}{13}x_3 &= \frac{85}{13} \qquad x_3 = \frac{85}{49} = 1.7347 \quad \blacktriangleleft \\ \frac{13}{2}x_2 - \frac{7}{2}\left(\frac{85}{49}\right) &= -\frac{27}{2} \qquad x_2 = -\frac{8}{7} = -1.1429 \quad \blacktriangleleft \\ 2x_1 - 3\left(-\frac{8}{7}\right) - \frac{85}{49} &= 3 \qquad x_1 = \frac{32}{49} = 0.6531 \quad \blacktriangleleft \end{aligned}$$

Problem 5

The augmented coefficient matrix is

$$[\mathbf{A}|\mathbf{B}] = \begin{bmatrix} 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 0 & 0 \end{bmatrix}$$

Before elimination, we exchange rows 2 and 3 in order to reduce the amount of algebra:

$$\begin{bmatrix} 2 & 0 & -1 & 0 & 1 & 0 \\ -1 & 2 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \end{bmatrix}$$

Elimination phase:

$$\text{row } 2 \leftarrow \text{row } 2 + \frac{1}{2} \times \text{row } 1$$

$$\begin{bmatrix} 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & -1/2 & 1 & 1/2 & 1 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \end{bmatrix}$$

$$\text{row } 3 \leftarrow \text{row } 3 - \frac{1}{2} \times \text{row } 2$$

$$\begin{bmatrix} 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & -1/2 & 1 & 1/2 & 1 \\ 0 & 0 & 9/4 & -1/2 & -1/4 & -1/2 \\ 0 & 0 & 1 & -2 & 0 & 0 \end{bmatrix}$$

$$\text{row } 4 \leftarrow \text{row } 4 - \frac{4}{9} \times \text{row } 3$$

$$\begin{bmatrix} 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & -1/2 & 1 & 1/2 & 1 \\ 0 & 0 & 9/4 & -1/2 & -1/4 & -1/2 \\ 0 & 0 & 0 & -16/9 & 1/9 & 2/9 \end{bmatrix}$$

First solution vector by back substitution:

$$\begin{aligned} -\frac{16}{9}x_4 &= \frac{1}{9} \qquad x_4 = -\frac{1}{16} \\ \frac{9}{4}x_3 - \frac{1}{2}\left(-\frac{1}{16}\right) &= -\frac{1}{4} \qquad x_3 = -\frac{1}{8} \\ 2x_2 - \frac{1}{2}\left(-\frac{1}{8}\right) + \left(-\frac{1}{16}\right) &= \frac{1}{2} \qquad x_2 = \frac{1}{4} \\ 2x_1 - \left(-\frac{1}{8}\right) &= 1 \qquad x_1 = \frac{7}{16} \end{aligned}$$

Second solution vector:

$$\begin{aligned}
-\frac{16}{9}x_4 &= \frac{2}{9} & x_4 &= -\frac{1}{8} \\
\frac{9}{4}x_3 - \frac{1}{2}\left(-\frac{1}{8}\right) &= -\frac{1}{2} & x_3 &= -\frac{1}{4} \\
2x_2 - \frac{1}{2}\left(-\frac{1}{4}\right) + \left(-\frac{1}{8}\right) &= 1 & x_2 &= \frac{1}{2} \\
2x_1 - \left(-\frac{1}{4}\right) &= 0 & x_1 &= -\frac{1}{8}
\end{aligned}$$

Therefore,

$$\mathbf{X} = \begin{bmatrix} 7/16 & -1/8 \\ 1/4 & 1/2 \\ -1/8 & -1/4 \\ -1/16 & -1/8 \end{bmatrix} \qquad \blacktriangleleft$$

Problem 6

After reordering rows, the augmented coefficient matrix is

$$\begin{bmatrix} 1 & 2 & 0 & -2 & 0 & -4 \\ 0 & 1 & -1 & 1 & -1 & -1 \\ 0 & 1 & 0 & 2 & -1 & 1 \\ 0 & 0 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & -1 & 1 & -2 \end{bmatrix}$$

Elimination phase:

$$\text{row } 3 \leftarrow \text{row } 3 - \text{row } 2$$

$$\begin{bmatrix} 1 & 2 & 0 & -2 & 0 & -4 \\ 0 & 1 & -1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & -1 & 1 & -2 \end{bmatrix}$$

$$\text{row } 4 \leftarrow \text{row } 4 - 2 \times \text{row } 3$$

$$\begin{bmatrix} 1 & 2 & 0 & -2 & 0 & -4 \\ 0 & 1 & -1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & -1 & 2 & -3 \\ 0 & 0 & 0 & -1 & 1 & -2 \end{bmatrix}$$

$$\text{row } 5 \leftarrow \text{row } 5 - \text{row } 4$$

$$\begin{bmatrix} 1 & 2 & 0 & -2 & 0 & -4 \\ 0 & 1 & -1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & -1 & 2 & -3 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Back substitution:

$$\begin{aligned} -x_5 &= 1 & x_5 = -1 \blacktriangleleft \\ -x_4 + 2(-1) &= -3 & x_4 = 1 \blacktriangleleft \\ x_3 + 1 &= 2 & x_3 = 1 \blacktriangleleft \\ x_2 - 1 + 1 - (-1) &= -1 & x_2 = -2 \blacktriangleleft \\ x_1 + 2(-2) - 2(1) &= -4 & x_1 = 2 \blacktriangleleft \end{aligned}$$

Problem 7

(a)

$$\mathbf{A} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$$

Use Gauss elimination storing each multiplier in the location occupied by the element that was eliminated (the multipliers are enclosed in boxes $\boxed{\text{thus}}$:

$$\text{row } 2 \leftarrow \text{row } 2 - \left(-\frac{1}{4}\right) \times \text{row } 2$$

$$\begin{bmatrix} 4 & -1 & 0 \\ \boxed{-1/4} & 15/4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$$

$$\text{row } 3 \leftarrow \text{row } 3 - \left(-\frac{4}{15}\right) \times \text{row } 2$$

$$\begin{bmatrix} 4 & -1 & 0 \\ \boxed{-1/4} & 15/4 & -1 \\ 0 & \boxed{-4/15} & 56/15 \end{bmatrix}$$

Thus

$$\mathbf{U} = \begin{bmatrix} 4 & -1 & 0 \\ 0 & 15/4 & -1 \\ 0 & 0 & 56/15 \end{bmatrix} \blacktriangleleft \qquad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -1/4 & 1 & 0 \\ 0 & -4/15 & 1 \end{bmatrix} \blacktriangleleft$$

(b)

$$\mathbf{A} = \mathbf{L}\mathbf{L}^T$$

Substituting for $\mathbf{L}\mathbf{L}^T$ from Eq. (2.16), we get

$$\begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} L_{11}^2 & L_{11}L_{21} & L_{11}L_{31} \\ L_{11}L_{21} & L_{21}^2 + L_{22}^2 & L_{21}L_{31} + L_{22}L_{32} \\ L_{11}L_{31} & L_{21}L_{31} + L_{22}L_{32} & L_{31}^2 + L_{32}^2 + L_{33}^2 \end{bmatrix}$$

Equating matrices term-by term:

$$\begin{aligned}
L_{11}^2 &= 4 \qquad L_{11} = 2 \\
2L_{21} &= -1 \qquad L_{21} = -\frac{1}{2} \\
2L_{31} &= 0 \qquad L_{31} = 0 \\
\left(-\frac{1}{2}\right)^2 + L_{22}^2 &= 4 \qquad L_{22} = \frac{\sqrt{15}}{2} \\
-\frac{1}{2}(0) + \frac{\sqrt{15}}{2}L_{32} &= -1 \qquad L_{32} = -\frac{2}{\sqrt{15}} \\
0^2 + \left(-\frac{2}{\sqrt{15}}\right)^2 + L_{33}^2 &= 4 \qquad L_{32} = 2\sqrt{\frac{14}{15}}
\end{aligned}$$

Therefore,

$$\mathbf{L} = \begin{bmatrix} 2 & 0 & 0 \\ -1/2 & \sqrt{15}/2 & 0 \\ 0 & -2/\sqrt{15} & 2\sqrt{14/15} \end{bmatrix} \blacktriangleleft$$

Problem 8

$$\mathbf{A} = \begin{bmatrix} -3 & 6 & -4 \\ 9 & -8 & 24 \\ -12 & 24 & -26 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -3 \\ 65 \\ -42 \end{bmatrix}$$

Decomposition of $\mathbf{A}$ (multipliers are enclosed in boxes):

$$\begin{aligned}
\text{row } 2 &\leftarrow \text{row } 2 - (-3) \times \text{row } 1 \\
\text{row } 3 &\leftarrow \text{row } 3 - 4 \times \text{row } 1
\end{aligned}$$

$$\begin{bmatrix} -3 & 6 & -4 \\ \boxed{-3} & 10 & 12 \\ \boxed{4} & 0 & -10 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} -3 & 6 & -4 \\ 0 & 10 & 12 \\ 0 & 0 & -10 \end{bmatrix} \qquad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

Solution of $\mathbf{Ly} = \mathbf{b}$:

$$\begin{aligned} y_1 &= -3 \\ -3(-3) + y_2 &= 65 \qquad y_2 = 56 \\ 4(-3) + y_3 &= -42 \qquad y_3 = -30 \end{aligned}$$

Solution of $\mathbf{Ux} = \mathbf{y}$:

$$\begin{aligned} -10x_3 &= -30 \qquad x_3 = 3 \blacktriangleleft \\ 10x_2 + 12(3) &= 56 \qquad x_2 = 2 \blacktriangleleft \\ -3x_1 + 6(2) - 4(3) &= -3 \qquad x_1 = 1 \blacktriangleleft \end{aligned}$$

Problem 9

$$\mathbf{A} = \begin{bmatrix} 2.34 & -4.10 & 1.78 \\ -1.98 & 3.47 & -2.22 \\ 2.36 & -15.17 & 6.18 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0.02 \\ -0.73 \\ -6.63 \end{bmatrix}$$

Decomposition of $\mathbf{A}$ (multipliers are enclosed in boxes):

$$\begin{aligned} \text{row } 2 &\leftarrow \text{row } 2 - (-0.846\,154) \times \text{row } 1 \\ \text{row } 3 &\leftarrow \text{row } 3 - 1.008\,547 \times \text{row } 1 \end{aligned}$$

$$\begin{bmatrix} 2.34 & -4.10 & 1.78 \\ \boxed{-0.846\,154} & 0.000\ 769 & -0.713\,846 \\ \boxed{1.008\,547} & -11.034\,96 & 4.384\,786 \end{bmatrix}$$

$$\text{row } 3 \leftarrow \text{row } 3 - (-14349.75) \times \text{row } 2$$

$$\begin{bmatrix} 2.34 & -4.10 & 1.78 \\ \boxed{-0.846\,154} & 0.000\ 769 & -0.713\,846 \\ \boxed{1.008\,547} & \boxed{-14349.75} & -10239.13 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} 2.34 & -4.10 & 1.78 \\ 0 & 0.000\ 769 & -0.713\,846 \\ 0 & 0 & -10\,239.1 \end{bmatrix} \qquad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -0.846\,154 & 1 & 0 \\ 1.008\,547 & -14349.7 & 1 \end{bmatrix}$$

Solution of $\mathbf{Ly} = \mathbf{b}$:

$$\begin{aligned} y_1 &= 0.02 \\ -0.846\,154(0.02) + y_2 &= -0.73 \qquad y_2 = -0.713\,07\,7 \\ 1.008\,547(0.02) - 14349.7(-0.713\,07\,7) + y_3 &= -6.63 \qquad y_3 = -10\,239.1 \end{aligned}$$

Solution of $\mathbf{Ux} = \mathbf{y}$:

$$\begin{aligned} -10\,239.1x_3 &= -10\,239.1 \qquad x_3 = 1.0 \blacktriangleleft \\ 0.000\ 769x_2 - 0.713\,846 &= -0.713\,07\,7 \qquad x_2 = 1.0 \blacktriangleleft \\ 2.34x_1 - 4.10 + 1.78 &= 0.02 \qquad x_1 = 1.0 \blacktriangleleft \end{aligned}$$

Problem 10

$$\mathbf{A} = \begin{bmatrix} 4 & -3 & 6 \\ 8 & -3 & 10 \\ -4 & 12 & -10 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Decomposition of **A** (multipliers are enclosed in boxes):

$$\begin{aligned} \text{row } 2 &\leftarrow \text{row } 2 - 2 \times \text{row } 1 \\ \text{row } 3 &\leftarrow \text{row } 3 - (-1) \times \text{row } 1 \end{aligned}$$

$$\begin{bmatrix} 4 & -3 & 6 \\ \boxed{2} & 3 & -2 \\ \boxed{-1} & 9 & -4 \end{bmatrix}$$

$$\text{row } 3 \leftarrow \text{row } 3 - 3 \times \text{row } 2$$

$$\begin{bmatrix} 4 & -3 & 6 \\ \boxed{2} & 3 & -2 \\ \boxed{-1} & \boxed{3} & 2 \end{bmatrix}$$

$$U = \begin{bmatrix} 4 & -3 & 6 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}$$

<u>First solution vector</u>
Solution of $\mathbf{Ly} = \mathbf{b}$:

$$\begin{aligned} y_1 &= 1 \\ 2(1) + y_2 &= 0 \qquad y_2 = -2 \\ -1 + 3(-2) + y_3 &= 0 \qquad y_3 = 7 \end{aligned}$$

Solution of $\mathbf{Uy} = \mathbf{x}$:

$$\begin{aligned} 2x_3 &= 7 \qquad x_3 = \frac{7}{2} \\ 3x_2 - 2\left(\frac{7}{2}\right) &= -2 \qquad x_2 = \frac{5}{3} \\ 4x_1 - 3\left(\frac{5}{3}\right) + 6\left(\frac{7}{2}\right) &= 1 \qquad x_1 = -\frac{15}{4} \end{aligned}$$

<u>Second solution vector</u>
Solution of $\mathbf{Ly} = \mathbf{b}$:

$$\begin{aligned} y_1 &= 0 \\ 2(0) + y_2 &= 1 \qquad y_2 = 1 \\ -1(0) + 3(1) + y_3 &= 0 \qquad y_3 = -3 \end{aligned}$$

Solution of $\mathbf{Ux} = \mathbf{y}$:

$$\begin{aligned} 2x_3 &= -3 \qquad x_3 = -\frac{3}{2} \\ 3x_2 - 2\left(-\frac{3}{2}\right) &= 1 \qquad x_3 = -\frac{2}{3} \\ 4x_1 - 3\left(-\frac{2}{3}\right) + 6\left(-\frac{3}{2}\right) &= 0 \qquad x_1 = \frac{7}{4} \end{aligned}$$

Therefore,

$$\mathbf{X} = \begin{bmatrix} 7/2 & -3/2 \\ 5/3 & -2/3 \\ -15/4 & 7/4 \end{bmatrix} \quad \blacktriangleleft$$

Problem 11

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 3/2 \\ 3 \end{bmatrix}$$

Substituting for $\mathbf{LL}^T$ from Eq. (2.16), we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} L_{11}^2 & L_{11}L_{21} & L_{11}L_{31} \\ L_{11}L_{21} & L_{21}^2 + L_{22}^2 & L_{21}L_{31} + L_{22}L_{32} \\ L_{11}L_{31} & L_{21}L_{31} + L_{22}L_{32} & L_{31}^2 + L_{32}^2 + L_{33}^2 \end{bmatrix}$$

Equating matrices term-by term:

$$\begin{aligned} L_{11} &= 1 \qquad L_{21} = 1 \qquad L_{31} = 1 \\ 1^2 + L_{22}^2 &= 2 \qquad L_{22} = 1 \\ (1)(1) + (1)L_{32} &= 2 \qquad L_{32} = 1 \\ 1^2 + 1^2 + L_{33}^2 &= 3 \qquad L_{33} = 1 \end{aligned}$$

Thus

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad \mathbf{L}^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution of $\mathbf{Ly} = \mathbf{b}$:

$$\begin{aligned} y_1 &= 1 \\ 1 + y_2 &= \frac{3}{2} \qquad y_2 = \frac{1}{2} \\ 1 + \frac{1}{2} + y_3 &= 3 \qquad y_3 = \frac{3}{2} \end{aligned}$$

Solution of $\mathbf{L}^T\mathbf{x} = \mathbf{y}$:

$$\begin{aligned} x_3 &= \frac{3}{2} \quad \blacktriangleleft \\ x_2 + \frac{3}{2} &= \frac{1}{2} \qquad x_2 = -1 \quad \blacktriangleleft \\ x_1 - 1 + \frac{3}{2} &= 1 \qquad x_1 = \frac{1}{2} \quad \blacktriangleleft \end{aligned}$$

Problem 12

$$A = \begin{bmatrix} 4 & -2 & -3 \\ 12 & 4 & -10 \\ -16 & 28 & 18 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1.1 \\ 0 \\ -2.3 \end{bmatrix}$$

Decomposition of $\mathbf{A}$ (multipliers are enclosed in boxes):

$$\begin{aligned} \text{row } 2 &\leftarrow \text{row } 2 - 3 \times \text{row } 1 \\ \text{row } 3 &\leftarrow \text{row } 3 - (-4) \times \text{row } 1 \end{aligned}$$

$$\begin{bmatrix} 4 & -2 & -3 \\ \boxed{3} & 10 & -1 \\ \boxed{-4} & 20 & 6 \end{bmatrix}$$

$$\text{row } 3 \leftarrow \text{row } 3 - 2 \times \text{row } 2$$

$$\begin{bmatrix} 4 & -2 & -3 \\ \boxed{3} & 10 & -1 \\ \boxed{-4} & \boxed{2} & 8 \end{bmatrix}$$

Therefore

$$\mathbf{U} = \begin{bmatrix} 4 & -2 & -3 \\ 0 & 10 & -1 \\ 0 & 0 & 8 \end{bmatrix} \qquad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -4 & 2 & 1 \end{bmatrix}$$

Solution of $\mathbf{Ly} = \mathbf{b}$:

$$\begin{aligned} y_1 &= 1.1 \\ 3(1.1) + y_2 &= 0 \qquad y_2 = -3.3 \\ -4(1.1) + 2(-3.3) + y_3 &= -2.3 \qquad y_3 = 8.7 \end{aligned}$$

Solution of $\mathbf{Ux} = \mathbf{y}$:

$$\begin{aligned} 8x_3 &= 8.7 \qquad x_3 = 1.0875 \quad \blacktriangleleft \\ 10x_2 - 1.0875 &= -3.3 \qquad x_2 = -0.221\,25 \quad \blacktriangleleft \\ 4x_1 - 2(-0.221\,25) - 3(1.0875) &= 1.1 \qquad x_1 = 0.98 \quad \blacktriangleleft \end{aligned}$$

Problem 13

$$\mathbf{A} = \begin{bmatrix} \alpha_1 & 0 & 0 & \cdots \\ 0 & \alpha_2 & 0 & \cdots \\ 0 & 0 & \alpha_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Since the banded structure of a matrix is preserved during decomposition, **L** must be a diagonal matrix. Therefore,

$$\mathbf{L}\mathbf{L}^T = \begin{bmatrix} L_{11}^2 & 0 & 0 & \cdots \\ 0 & L_{22}^2 & 0 & \cdots \\ 0 & 0 & L_{33}^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

It follows from $\mathbf{A} = \mathbf{L}\mathbf{L}^T$ that

$$\mathbf{L} = \begin{bmatrix} \sqrt{\alpha_1} & 0 & 0 & \cdots \\ 0 & \sqrt{\alpha_2} & 0 & \cdots \\ 0 & 0 & \sqrt{\alpha_3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \blacktriangleleft$$

Problem 14

```
function [X,det] = gauss2(A,B)
% Solves A*X = B by Gauss elimination and computes det(A).
% USAGE: [X,det] = gauss(A,B)

[n,m] = size(B);
for k = 1:n-1                   % Elimination phase
    for i= k+1:n
        if A(i,k) ~= 0
            lambda = A(i,k)/A(k,k);
            A(i,k+1:n) = A(i,k+1:n) - lambda*A(k,k+1:n);
            B(i,:)= B(i,:) - lambda*B(k,:);
         end
     end
end
if nargout == 2; det = prod(diag(A)); end
for k = n:-1:1                  % Back substitution phase
    for i = 1:m
        B(k,i) = (B(k,i) - A(k,k+1:n)*B(k+1:n,i))/A(k,k);
```

```
    end
end
X = B;
```

Testing gauss2:

```
>> A = [2 -1 0; -1 2 -1; 0 -1 1];
>> B = [1 0 0; 0 1 0; 0 0 1];
>> [X,detA] = gauss2(A,B)
X =
    1.0000    1.0000    1.0000
    1.0000    2.0000    2.0000
    1.0000    2.0000    3.0000
detA =
     1
```

Problem 15

```
function hilbert(n)
% Solves A*x = b by LU decomposition, where
% [A] is an n x n Hilbert matrix and b(i) is the sum of
% the elemnts in the ith row of [A].
% USAGE: hibert(n)

A = zeros(n); b = zeros(n,1);
for i = 1:n
    for j = 1:n
        A(i,j)= 1/(i + j - 1);
        b(i) = b(i) + A(i,j);
    end
end
A = LUdec(A); x = LUsol(A,b)
```

The largest n for which 6-figure accuracy is achieved seems to be 8:

```
>> format long
>> hilbert(8)
x =
   0.99999999996352
   1.00000000197488
   0.99999997404934
   1.00000014104434
   0.99999961908263
```

```
1.00000054026798
0.99999961479776
1.00000010884699
```

Problem 16

Forward sustitution The kth equation of $\mathbf{Ly} = \mathbf{b}$ is

$$L_{k1}y_1 + L_{k2}y_2 + \cdots + L_{kk}y_k = b_k$$

Solving for y_k yields

$$\begin{aligned} y_k &= \frac{b_k - (L_{k,1}y_1 + L_{k,2}y_2 + \cdots + L_{k,k-1}y_{k-1})}{L_{kk}} \\ &= b_k - \frac{\left[\begin{array}{cccc} L_{k,1} & L_{k,2} & \cdots & L_{k,k-1} \end{array}\right] \cdot \left[\begin{array}{cccc} y_1 & y_2 & \cdots & y_{k-1} \end{array}\right]}{L_{k,k}} \end{aligned}$$

This expression, evaluated with $k = 1, 2, \ldots, n$ (in that order), constitutes the forward substitution phase In `choleskiSol` the b's are overwritten with y's during the computations.

Back substitution A typical (kth) equation of $\mathbf{L}^T\mathbf{x} = \mathbf{y}$ is

$$L_{k,k}x_k + L_{k+1,k}x_{k+1} + L_{k+2,k}x_{k+2} + \cdots + L_{n,k}x_n = y_k$$

The solution for x_k is

$$\begin{aligned} x_k &= \frac{y_k - (L_{k+1,k}x_{k+1} + L_{k+2,k}x_{k+2} + \ldots L_{n,k}x_n)}{L_{k,k}} \\ &= \frac{y_k - \left[\begin{array}{cccc} L_{k+1,k} & L_{k+2,k} & \cdots & L_{n,k} \end{array}\right] \cdot \left[\begin{array}{cccc} x_{k+1} & x_{k+2} & \cdots & x_n \end{array}\right]}{L_{kk}} \end{aligned}$$

In back substitution we evaluate this expression in the order $k = n, n-1, \ldots, 1$. Note that in `choleskiSol` the vector $\mathbf{x}$ overwrites the vector $\mathbf{y}$.

Problem 17

```
% problem2_1_17
x = [0 1 3 4]'; y = [10 35 31 2]';
n = length(x);
A = zeros(n);
```

```
for i = 1:n; A(:,i) = x.^(i-1); end
L = LUdec(A);
coefficients = LUsol(L,y)

>> coefficients =
    10
    34
    -9
     0
```

Problem 18

```
% problem2_1_18
x = [0 1 3 5 6]'; y = [-1 1 3 2 -2]';
n = length(x);
A = zeros(n);
for i = 1:n; A(:,i) = x.^(i-1); end
L = LUdec(A);
coefficients = LUsol(L,y)

>> coefficients =
  -1.00000000000000
   2.68333333333333
  -0.87500000000000
   0.21666666666667
  -0.02500000000000
```

Problem 19

$$
\begin{aligned}
f(x) &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 \\
f''(x) &= 2c_2 + 6c_3 x + 12c_4 x^2
\end{aligned}
$$

The specified conditions result in the equations

$$\mathbf{Ac} = \mathbf{y}$$

```
% problem2_1_19
A = zeros(5);
A(1,:) = [1 0 0 0 0];                    % f(0)
```

```
A(2,:) = [1 0.75 0.75^2 0.75^3 0.75^4];  % f(0.75)
A(3,:) = [1 1 1 1 1];                    % f(1)
A(4,:) = [0 0 2 0 0];                    % f''(0)
A(5,:) = [0 0 2 6 12];                   % f''(1)
y = [1 -0.25 1 0 0]';
L = LUdec(A);
c = LUsol(L,y)

>> c =
   1.00000000000000
  -5.61403508771930
   0.00000000000000
  11.22807017543859
  -5.61403508771930
```

Therefore, the polynomial is

$$f(x) = 1 - 5.6140x + 11.2281x^3 - 5.6140x^4$$

Problem 20

$$\mathbf{A} = \begin{bmatrix} 3.50 & 2.77 & -0.76 & 1.80 \\ -1.80 & 2.68 & 3.44 & -0.09 \\ 0.27 & 5.07 & 6.90 & 1.61 \\ 1.71 & 5.45 & 2.68 & 1.71 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 7.31 \\ 4.23 \\ 13.85 \\ 11.55 \end{bmatrix}$$

```
% problem2_1_20
A = [3.50 2.77 -0.76  1.80
    -1.80 2.68  3.44 -0.09
     0.27 5.07  6.90  1.61
     1.71 5.45  2.68  1.71];
b = [7.31 4.23 13.85 11.55]';
n = length(b);
L = LUdec(A);
x = LUsol(L,b)
detA = prod(diag(L))
Ax = A*x

>> x =
   1.00000000000011
   1.00000000000002
   1.00000000000003
   0.99999999999976
```

```
detA =
  -0.22579734000001
Ax =
   7.31000000000000
   4.23000000000000
  13.85000000000000
  11.55000000000000
```

The determinant is a little smaller than the elements of **A**, indicating a mild case of ill-conditioning. From the results it appears that the solution is 12-figure accurate.

Problem 21

```
>> A = [1 -1 -1; 0 1 -2; 0 0 1];
>> inv(A)

ans =

     1     1     3
     0     1     2
     0     0     1
```

(a)

$$\begin{aligned} \|A\|_e &= \sqrt{1^2+1^2+1^2+1^2+2^2+1^2} = 3 \\ \left\|A^{-1}\right\|_e &= \sqrt{1^2+1^2+3^2+1^2+2^2+1^2} = \sqrt{17} \\ \mathrm{cond}_e &= \|A\|_e \left\|A^{-1}\right\|_e = 3\sqrt{17} = 12.37 \quad \blacktriangleleft \end{aligned}$$

(b)

$$\begin{aligned} \|A\|_\infty &= 3 \text{ (determined by row 1 or row 2)} \\ \left\|A^{-1}\right\|_\infty &= 5 \text{ (determined by row 1)} \\ \mathrm{cond}_\infty &= \|A\|_\infty \left\|A^{-1}\right\|_\infty = 3(5) = 15 \quad \blacktriangleleft \end{aligned}$$

Problem 22

```
function eCond = cond(A)
% Returns condition number of [A]
```

```
eCond = norm(A)*norm(inv(A));

function eNorm = norm(A)
% Returns euclidean norm of [A]
n = size(A,1);
eNorm = 0;
for i = 1:n
    eNorm = eNorm + sum(A(i,:).^2);
end
eNorm = sqrt(eNorm);

>> A = [ 1  4  9 16
         4  9 16 25
         9 16 25 36
        16 25 36 49];

>> cond(A)
Warning: Matrix is close to singular or badly scaled.

ans =
  3.0371e+016
```

Problem 23

```
% problem2_1_23
A = rand(500);
b = sum(A,2);
x = gauss(A,b)

x =
    1.0000
    1.0000
    1.0000
```

⋮

Gauss elimination is remarkably stable—there is no significant rounoff error in the solution of 500 equations. We also solved 1000 equations with the same result.

Problem 24

```
% problem2_1_24
A= [ 5+1j  5+2j -5+3j  6-3j
     5+2j  7-2j  8-1j -1+3j
    -5+3j  8-1j -3-3j  2+2j
     6-3j -1+3j, 2+2j  0+8j];
b = [15-35j; 2+10j; -2-34j; 8+14j];
x = gauss(A,b)

x =
   2.0000 - 0.0000i
  -0.0000 - 4.0000i
   0.0000 + 4.0000i
   1.0000 + 1.0000i
```

Problem 25

```
% problem2_1_25
theta = pi/4.0
g = 9.81
m = [ 10.0, 4.0,  5.0, 6.0]';
mu = [0.25, 0.3,  0.2]';
A = [ 1.0,  0.0,  0.0, m(1)
     -1.0,  1.0,  0.0, m(2)
      0.0, -1.0,  1.0, m(3)
      0.0,  0.0, -1.0, m(4)];
b = zeros(4,1);
for i = 1:3
    b(i) = m(i)*g*(sin(theta) - mu(i)*cos(theta));
end
b(4) = -m(4)*g;
x = gauss(A,b)

x =
   35.8914
   48.8606
   68.5404
    1.6134
```

Hence $T_1 = 35.89$ N, $T_2 = 48.86$ N, $T_3 = 68.54$ N and $a = 1.6134$ m/s^2 ◀