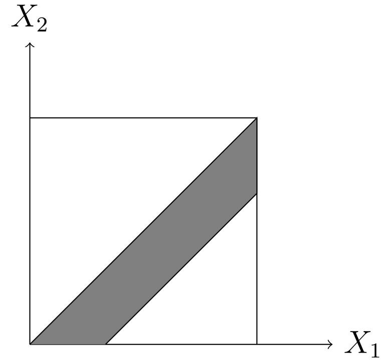
1 Introduction and objectives

Solution 1. (Probabilities of basic events)

In each case, the shaded region represents the (X_1, X_2) *values satisfying the corresponding inequalities. Since* X_1 *and* X_2 *are independent and uniformly distributed, the area of the shaded region gives the probability of the inequality being satisfied. We use* $Pr\{\cdot\}$ *to denote the probability of an event.*

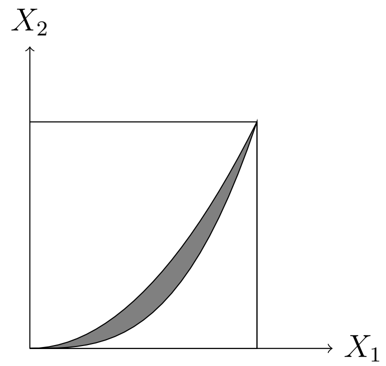
(a)

$$Pr\left\{0 \le X_1 - X_2 \le \frac{1}{3}\right\} = \frac{1}{2} - \frac{1}{2} \times \left(\frac{2}{3} \times \frac{2}{3}\right) = \frac{5}{18}.$$



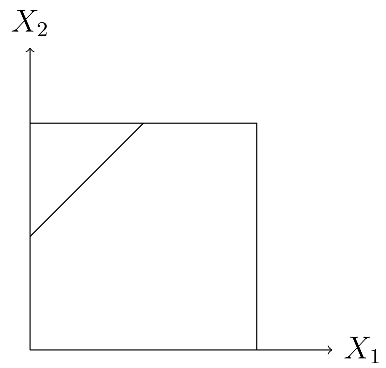
(b)

$$Pr\left\{X_1^3 \le X_2 \le X_1^2\right\} = \int_0^1 (x^2 - x^3)\, dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{12}.$$



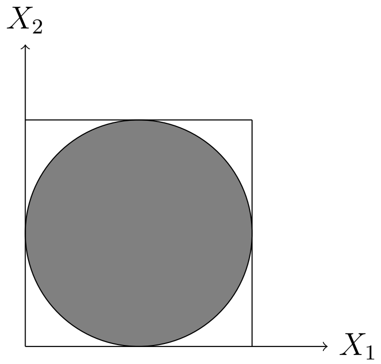
(c)

$$Pr\left\{X_2 - X_1 = \frac{1}{2}\right\} = 0.$$



(d)

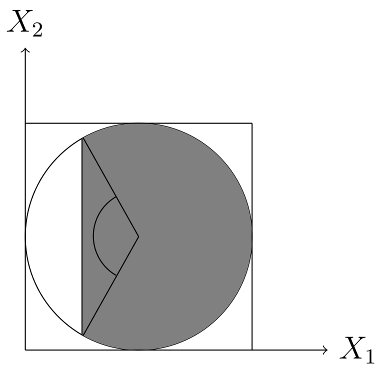
$$Pr\left\{\left(X_1-\frac{1}{2}\right)^2+\left(X_2-\frac{1}{2}\right)^2\le\left(\frac{1}{2}\right)^2\right\}=\pi\left(\frac{1}{2}\right)^2=\frac{\pi}{4}.$$



(e) In this part we have

$$\begin{aligned}&Pr\left\{\left(X_1-\frac{1}{2}\right)^2+\left(X_2-\frac{1}{2}\right)^2\le\left(\frac{1}{2}\right)^2\;\middle|\;X_1\ge\frac{1}{4}\right\}\\&=\frac{Pr\left\{\left(X_1-\frac{1}{2}\right)^2+\left(X_2-\frac{1}{2}\right)^2\le\left(\frac{1}{2}\right)^2\;,\;X_1\ge\frac{1}{4}\right\}}{Pr\left\{X_1\ge\frac{1}{4}\right\}}\\&=\frac{\frac{\pi}{6}+\frac{\sqrt{3}}{16}}{\frac{3}{4}}.\end{aligned}$$

It can easily be seen that the probability term in the numerator is equal to the area of the shaded region in the figure below. We can divide the shaded area into two parts, triangular and sub circular. It is easy to show that the angle of the triangle on the picture is 120° *so the sub circular part consists of* $\frac{2}{3}$ *of the circle area. So the sub circular part's area is* $\frac{2}{3}\,\pi(\frac{1}{2})^2=\frac{\pi}{6}$ *and the triangular part's area is* $\frac{\sqrt{3}}{16}$*. Summing the area of these two parts, we reach the final result.*



Solution 2. (Basic probabilities)

(a) *First, we find the probability of the complement of the event, namely the probability of drawing only black balls. This probability is equal to*

$$Pr\{All\ k\ balls\ are\ black\} = \frac{\binom{n}{k}}{\binom{m+n}{k}}.$$

Therefore the probability of drawing at least one white ball is equal to

$$Pr\{At\ least\ one\ ball\ is\ white\} = 1 - \frac{\binom{n}{k}}{\binom{m+n}{k}}.$$

(b) *Define the following random variables*

$$X = \begin{cases} 0 & \text{if the chosen coin is fair,} \\ 1 & \text{otherwise,} \end{cases}$$

and

$$Y = \begin{cases} 00 & \text{if both outcomes are tail,} \\ 01 & \text{if the first one is tail, the second one is head,} \\ 10 & \text{if the first one is head, the second one is tail,} \\ 11 & \text{if both outcomes are head.} \end{cases}$$

So having these two random variables defined, we want to compute $Pr\{X = 0|Y = 11\}$*. So we can write*

$$\begin{aligned} Pr\{X = 0|Y = 11\} &= \frac{Pr\{Y = 11|X = 0\}Pr\{X = 0\}}{Pr\{Y = 11\}} \\ &= \frac{1/4 \times 1/2}{Pr\{Y = 11\}} \\ &= \frac{1/8}{Pr\{Y = 11\}}. \end{aligned}$$

Then for $Pr\{Y = 11\}$ *we have*

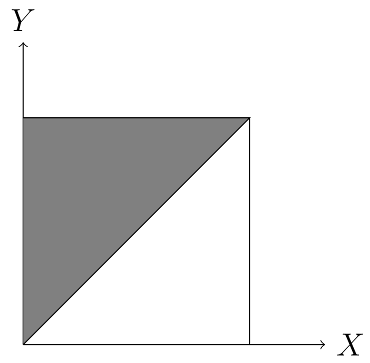
$$\begin{aligned} Pr\{Y = 11\} &= Pr\{X = 0\} \cdot Pr\{Y = 11|X = 0\} + Pr\{X = 1\} \cdot Pr\{Y = 11|X = 1\} \\ &= 1/2 \times 1/4 + 1/2 \times 1 \\ &= 5/8. \end{aligned}$$

So, finally we have

$$Pr\{X = 0|Y = 11\} = \frac{1/8}{5/8} = \frac{1}{5}.$$

Solution 3. (Conditional distribution)

The probability mass has been distributed uniformly on the upper triangular area according to the shape below:



(a) *If X and Y were independent then the distribution of X would not depend on Y. This is clearly not the case. In fact, the range of values taken by X is between 0 and Y.*

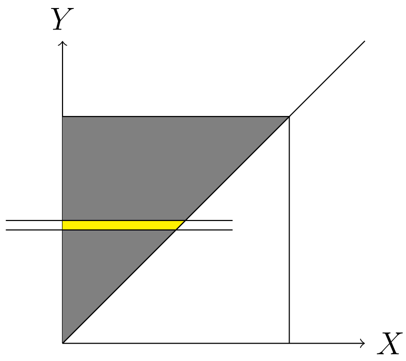
(b) *The integral of $f_{X,Y}(x, y)$ must be 1. Hence $A \times \frac{1}{2} = 1$ and so $A = 2$.*

(c) *We know that $f_Y(y)\, dy = Pr\{y < Y < y + dy\}$, but for a special y as can be seen from the figure below, this probability mass is equal to A times the area of a rectangle with length y and width dy when $0 \le y \le 1$.*

$$f_Y(y) = \begin{cases} 2y & 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Or more formally

$$f_Y(y) = \int_0^1 f_{X,Y}(x, y)\, dx = \int_0^y 2\, dx = 2y.$$



(d) *Under the condition $Y = y$, the random variable X is uniformly distributed between 0 and y and so $f(y) = \mathbb{E}[X|Y = y] = \frac{y}{2}$.*

(e) *$f(Y)$ is a function of Y so it is a random variable and we can compute its expected value.*

$$\mathbb{E}[f(Y)] = \int_0^1 f(y) f_Y(y)\, dy = \int_0^1 y^2\, dy = \frac{1}{3}.$$

(f) *We compute $\mathbb{E}[X]$ using the definition.*

$$\mathbb{E}[X] = \iint x f_{X,Y}(x, y)\, dx\, dy = \int_0^1 \left[\int_0^y 2x\, dx\right] dy = \frac{1}{3},$$

and it is seen that $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]]$. This result, which holds in general, is named the law of total expectation.

Solution 4. (Playing darts)

(a) $X = ZX_1 + (1 - Z)X_2$.

(b) *Note that $\mathbb{E}[X]=0$, because expectation is linear and Z is independent from X_1 and X_2. Thus,*

$$\begin{aligned}\mathrm{Var}(X) &= \mathbb{E}\left[X^2\right]-\mathbb{E}[X]^2\\ &= \mathbb{E}\left[X^2\right]\\ &= \mathbb{E}\left[X^2|Z=1\right]p+\mathbb{E}\left[X^2|Z=0\right](1-p)\\ &= p\sigma_1^2+(1-p)\sigma_2^2.\end{aligned}$$

X is not Gaussian. In fact X is not a linear combination of two Gaussians, it is rather a mixture *of two Gaussians. One can use the characteristic function to show rigorously that X is not a Gaussian, but this is outside the scope of this class.*

(c)

$$\begin{aligned}\mathbb{E}[S] &= p\int_{-\infty}^{\infty}|x|\frac{1}{\sigma_1\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma_1^2}}\,dx+(1-p)\int_{-\infty}^{\infty}|x|\frac{1}{\sigma_2\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma_2^2}}\,dx\\ &= 2p\int_0^{\infty}x\frac{1}{\sigma_1\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma_1^2}}\,dx+2(1-p)\int_0^{\infty}x\frac{1}{\sigma_2\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma_2^2}}\,dx.\end{aligned}$$

With the change of variables $u_1=\frac{x^2}{2\sigma_1^2}$ and $u_2=\frac{x^2}{2\sigma_2^2}$, we obtain

$$\begin{aligned}\mathbb{E}[S] &= 2p\frac{\sigma_1}{\sqrt{2\pi}}\int_0^{\infty}e^{-u_1}\,du_1+2(1-p)\frac{\sigma_2}{\sqrt{2\pi}}\int_0^{\infty}e^{-u_2}\,du_2\\ &= \frac{2}{\sqrt{2\pi}}\left[p\sigma_1+(1-p)\sigma_2\right].\end{aligned}$$

Solution 5. (Uncorrelated vs. independent random variables)
Note:

- *By definition, X and Y are uncorrelated if and only if*

$$0=\mathrm{cov}(X,Y)=\mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])]=\mathbb{E}[XY]-\mathbb{E}[X]\mathbb{E}[Y].$$

  Hence $\mathrm{cov}(X,Y)=0$ is equivalent to the the condition $\mathbb{E}[XY]=\mathbb{E}[X]\mathbb{E}[Y]$.

- *X and Y are independent when $f_{XY}=f_Xf_Y$.*

(a) *Assume that the random variables X and Y are independent. Then*

$$\begin{aligned}\mathbb{E}[XY] &= \iint xyf_{X,Y}(x,y)\,dx\,dy=\iint xyf_X(x)f_Y(y)\,dx\,dy\\ &= \int xf_X(x)\,dx\int yf_Y(y)\,dy=\mathbb{E}[X]\mathbb{E}[Y],\end{aligned}$$

where the second equality follows from the assumption that X and Y are independent. Hence, if X and Y are independent, they are also uncorrelated.

(b) *X and Y are obviously dependent. For example, $X = 0$ implies $U = 0$ and $V = 0$. Hence it implies also $Y = 0$. The marginals of X and Y are*

$$X = \begin{cases} 0 & \text{with prob. } \frac{1}{4}, \\ 1 & \text{with prob. } \frac{1}{2}, \\ 2 & \text{with prob. } \frac{1}{4}, \end{cases}$$

$$Y = \begin{cases} 0 & \text{with prob. } \frac{1}{2}, \\ 1 & \text{with prob. } \frac{1}{2}. \end{cases}$$

The mean for X is $\mathbb{E}[X] = 1$ and for Y it is $\mathbb{E}[Y] = \frac{1}{2}$. Finally, we have that

$$\mathbb{E}[XY] = \left(\frac{1}{4} \times 0 \times 0\right) + \left(\frac{1}{4} \times 1 \times 1\right) + \left(\frac{1}{4} \times 1 \times 1\right) + \left(\frac{1}{4} \times 0 \times 2\right) = \frac{1}{2}.$$

From the above we obtain

$$\text{cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0.$$

Therefore, we see that X and Y are uncorrelated, even though they are dependent.

Solution 6. (Monty Hall)

(a) *$Pr\{A$ contains one million Swiss francs$\} = 1/3$.*

(b) *Observe that B contains the money if and only if A does not contain the money, thus*

$$Pr\{B \text{ contains one million Swiss francs}\} = Pr\{A \text{ contains nothing}\} = 2/3.$$

(c) *A reasonable person will choose B since it has a larger probability of containing the money.*

2 Receiver design for discrete-time observations: First layer

Solution 1. (Hypothesis testing: Uniform and uniform)

(a) *Let $l(y)$ be the number of 0's in the sequence y.*

$$P_{Y|H}(y|0) = \frac{1}{2^{2k}}$$

$$P_{Y|H}(y|1) = \begin{cases} \frac{1}{\binom{2k}{k}}, & \text{if } l = k \\ 0, & \text{otherwise} \end{cases}$$

(b) *The ML decision rule is:*

$$P_{Y|H}(y|1) \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} P_{Y|H}(y|0)$$

Because $\frac{1}{\binom{2k}{k}} > \frac{1}{2^{2k}}$ for any value of k, the ML decision rule becomes

$$\hat{H} = \begin{cases} 0, & \text{if } l(y) \neq k \\ 1, & \text{if } l(y) = k. \end{cases}$$

The single number needed is $l(y)$, the number of 0's in the sequence y.

(c) *The decision rule that minimizes the error probability is the MAP rule:*

$$P_{Y|H}(y|1)P_H(1) \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} P_{Y|H}(y|0)P_H(0).$$

The MAP decision rule gives $\hat{H} = 0$ whenever $l(y) \neq k$. When $l(y) = k$:

$$\hat{H} = \begin{cases} 0, & \text{if } \frac{\binom{2k}{k}}{2^{2k}} \geq \frac{P_H(1)}{P_H(0)} \\ 1, & \text{otherwise.} \end{cases}$$

(d) *Trivial solution: If $P_H(1) = 1$ then $\hat{H} = 1$ for all y (In this case, $l(y) = k$ is guaranteed). Similarly, if $P_H(0) = 1$ then $\hat{H} = 0$ for all y.*

Now assume $P_H(1) \neq 1$. Then there is a nonzero probability that $l(y) \neq k$, in which case $\hat{H} = 0$. The MAP decision rule always chooses $\hat{H} = 0$ if

$$\frac{\binom{2k}{k}}{2^{2k}} \geq \frac{P_H(1)}{P_H(0)} \iff P_H(0) \geq \frac{\frac{1}{\binom{2k}{k}}}{\frac{1}{\binom{2k}{k}} + \frac{1}{2^{2k}}}.$$

Solution 2. (The "Wetterfrosch")