CHAPTER 1

THE PHYSICS OF WAVES

Problem 1.1 Show that

$$\Psi(x,t) = (x - vt)^2$$

is a traveling wave.

Show that $\Psi(x,t)$ is a wave by substitution into Equation 1.1. Proceed as in Example 1.1.

On-line version uses $\Psi(x,t) = a\,(x - bt)^2$ where a and b are randomized integers. All intermediate steps are entered as symbolic expressions as in Example 1.1.

Using a and b as parameters, correct answers are as follows:

$$\frac{\partial \Psi}{\partial t} = -2ba\,(x - bt)$$

$$\frac{\partial \Psi^2}{\partial t^2} = 2a\,(b)^2$$

$$\frac{\partial \Psi}{\partial x} = 2a\,(x - bt)$$

$$\frac{\partial \Psi^2}{\partial x^2} = 2a$$

Identifying b as the velocity gives

$$\frac{1}{v^2}\frac{\partial \Psi^2}{\partial t^2} = 2a$$

Specific answers depend on the particular values of the randomized parameters a and b. The symbolic equation parser *grades algebraically equivalent answers as correct*.

Problem 1.2 A $1.00\,m$ long string is stretched to a tension of $100\,N$. Find the wave speed if it has a mass of $0.100\,g$.

Using

$$v = \sqrt{\frac{F}{\mu}}$$

with $F = 100N$ and $\mu = 10^{-4}kg/m$ gives $v = 1000m/s$.

On-line version randomizes the string length, tension and string mass.

Problem 1.3 Show that

$$\Psi(x,t) = Ae^{-(a^2x^2 + b^2t^2 + 2\,a\,b\,x\,t)}$$

is a traveling wave, and find the wave speed and direction of propagation. Assume that A, a and b are all constants, and that a and b have units that make the quantity in the exponential function unitless.

Substitution into the differential wave equation is tedious. The easier approach is to recast into the form of Equations 1.2 and 1.2. For the above wavefuntion, this gives

$$\Psi = \Psi\left(x + vt\right) = Ae^{-(ax+bt)^2} = Ae^{-a^2\left(x+\frac{b}{a}t\right)^2}$$

On-line version Assigns integer values to a an b, and asks for a symbolic expression for velocity. The correct answer is entered symbolically as $-\frac{b}{a}$. Specific answers depend on the particular values of the randomized parameters a and b. The symbolic equation parser *grades algebraically equivalent answers as correct*.

Problem 1.4 Show that $g(x + vt)$ in Equation 1.3 is a solution of the one-dimensional differential wave equation.

Substitute into the differential wave equation as in Section 1.3 for $f(x - vt)$.

On-line version uses symbolic input to check the needed derivatives. Letting $u = x + vt$, the following derivatives are entered as indicated on the right hand side of the following equations:

$$\frac{\partial g}{\partial t} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial t} = \frac{\partial g}{\partial u}v$$
$$\frac{\partial^2 g}{\partial t^2} = \frac{\partial^2 g}{\partial u^2}v^2$$
$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial u}$$
$$\frac{\partial^2 g}{\partial x^2} = \frac{\partial^2 g}{\partial u^2}$$
$$\frac{1}{v^2}\frac{\partial^2 g}{\partial t^2} = \frac{\partial^2 g}{\partial u^2}$$

Since the last two results agree, $g(x + vt)$ is a traveling wave.

Problem 1.5 Show that $g(x + vt)$ in Equation 1.3 represents a wave that travels in the negative-x direction.

Let $u = x + vt$. A particular point on the wave (eg. the crest) is described by a constant value of u. Thus

$$du = dx + vdt = 0$$

giving

$$\frac{dx}{dt} = -v$$

Thus, $g(x + vt)$ represents a backward traveling wave.

Problem 1.6 Show explicitly (by direct substitution) that the function

$$\Psi(x, t) = A \sin \frac{2\pi}{\lambda}(x \mp vt)$$

solves the one-dimensional wave equation.

Substitute $\Psi(x, t)$ into the differential wave equation.

On-line version uses symbolic input to check the needed derivatives for the specific case of $\Psi(x, t) = A \sin \frac{2\pi}{\lambda}(x - vt)$. The following derivatives are entered as indicated on the right hand side of the following equations:

$$\frac{\partial \Psi}{\partial t}(x,t) = -v\frac{2\pi}{\lambda}A\cos\left(\frac{2\pi}{\lambda}(x-vt)\right)$$
$$\frac{\partial^2 \Psi}{\partial t^2} = -v^2\left(\frac{2\pi}{\lambda}\right)^2 A\sin\left(\frac{2\pi}{\lambda}(x-vt)\right)$$
$$\frac{\partial \Psi}{\partial x} = \frac{2\pi}{\lambda}A\cos\left(\frac{2\pi}{\lambda}(x-vt)\right)$$
$$\frac{\partial^2 \Psi}{\partial x^2} = -\left(\frac{2\pi}{\lambda}\right)^2 A\sin\left(\frac{2\pi}{\lambda}(x-vt)\right)$$
$$\frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2} = -\left(\frac{2\pi}{\lambda}\right)^2 A\sin\left(\frac{2\pi}{\lambda}(x-vt)\right)$$

Since the last two results agree, $\Psi(x,t)$ is a traveling wave.

Problem 1.7 Show explicitly that the function

$$\Psi(x,t) = A\cos(kx \mp \omega t + \phi)$$

solves the one-dimensional wave equation.

Substitute $\Psi(x,t)$ into the differential wave equation.

On-line version uses symbolic input to check the needed derivatives for the specific case of $\Psi(x,t) = A\cos(kx + \omega t + \phi)$ The following derivatives are entered as indicated on the right hand side of the following equations:

$$\frac{\partial \Psi}{\partial t} = -\omega A\sin(kx + \omega t + \varphi)$$
$$\frac{\partial^2 \Psi}{\partial t^2} = -\omega^2 A\cos(kx + \omega t + \varphi)$$
$$\frac{\partial \Psi}{\partial x} = -kA\sin(kx + \omega t + \varphi)$$
$$\frac{\partial^2 \Psi}{\partial x^2} = -k^2 A\cos(kx + \omega t + \varphi)$$
$$\frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2} = -k^2 A\cos(kx + \omega t + \varphi)$$

where the last result uses $v = \frac{\omega}{k}$. Since the last two results agree, $\Psi(x,t)$ is a traveling wave.

Problem 1.8 The speed of sound is $343\,m/s$ in air and $1493\,m/s$ in water. Find the wavelength emitted by a transducer that oscillates at 1000 Hz in both air and water. Assume that the frequency of the oscillator is the same in both media. In each case, find k and ω.

The angular frequency $\omega = 2\pi f$ is the same for each case, but the wavenumber $k = \omega/v$ differs. Using the book parameters, $\omega = 6283\,rad/s$. When $v = 343\,m/s$, $k = 18.3\,m^{-1}$, and when $v = 1493\,m/s$, $k = 4.21\,m^{-1}$.

On-line version randomizes the frequency.

Problem 1.9 A harmonic traveling wave is given by $\Psi(z,t) = A\sin(50z + 3000t)$. Find the wave speed, frequency, angular frequency, and direction of propagation.

By inspection, $\omega = 50\,m^{-1}$ and $\omega = 3000\,rad/s$, and the wave moves in the negative z direction. The wave speed is $v = \omega/k = 60\,m/s$ and the frequency is $f = \omega/2\pi = 477\,Hz$.

On-line version randomizes amplitude A (and asks for it), k and ω.

Problem 1.10 Show explicitly that the function

$$\Psi(x,t) = A\sin(kx - \omega t + \phi) + B\sin(kx + \omega t + \phi)$$

solves the one-dimensional wave equation.

Verify that $\Psi(x,t)$ satisfies the differential wave equation by computing the necessary derivatives:

$$\begin{aligned}
\frac{\partial\Psi}{\partial t} &= -\omega A\cos(kx - \omega t + \varphi) + \omega A\cos(kx + \omega t + \varphi) \\
\frac{\partial^2\Psi}{\partial t^2} &= -\omega^2 A\sin(kx - \omega t + \varphi) - \omega^2 A\sin(kx + \omega t + \varphi) = -\omega^2\Psi \\
\frac{\partial\Psi}{\partial x} &= kA\cos(kx - \omega t + \varphi) + kA\cos(kx + \omega t + \varphi) \\
\frac{\partial^2\Psi}{\partial x^2} &= -k^2\Psi \\
\frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2} &= -\frac{\omega^2}{v^2}\Psi = -k^2\Psi = \frac{\partial^2\Psi}{\partial x^2}
\end{aligned}$$

Problem 1.11 Show the *small angle approximation*: for small θ, $\sin\theta \approx \theta$. Check the approximation for $\theta = 5°$, $10°$, and $20°$.

For each case, calculate the angle in radians along with the sine.
$\theta = 5° = 0.0872\,rad; \sin(5°) = 0.0872$
$\theta = 10° = 0.175\,rad; \sin(10°) = 0.174$

$\theta = 20^\circ = 0.349\,rad; \sin(20^\circ) = 0.342$

On-line version randomizes each of the above angles in the vicinity of the book values.

Problem 1.12 Show that $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$

Proceed as in Example 1.3:

$$f|_0 = 1$$

$$\frac{d}{dx}\cos x = -\sin x \Rightarrow \left.\frac{df}{dx}\right|_0 = 0$$

$$\left.\frac{d^2f}{dx^2}\right|_0 = -1 \quad \left.\frac{d^3f}{dx^3}\right|_0 = 0 \quad \left.\frac{d^4f}{dx^4}\right|_0 = +1 \quad \left.\frac{d^5f}{dx^5}\right|_0 = 0 \quad \left.\frac{d^6f}{dx^6}\right|_0 = -1$$

Substitution into the Taylor Series formula gives the desired result.

Problem 1.13 Show that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$

Proceed as in Example 1.3:

$$f|_0 = \left.\frac{df}{dx}\right|_0 = \left.\frac{d^2f}{dx^2}\right|_0 = \left.\frac{d^3f}{dx^3}\right| = \left.\frac{d^4f}{dx^4}\right|_0 = \left.\frac{d^5f}{dx^5}\right|_0 = \left.\frac{d^6f}{dx^6}\right|_0 = 1$$

Substitution into the Taylor Series formula gives the desired result.

Problem 1.14 Express the following numbers in polar form: $4 + 5i$, $-4 - 5i$, $4 - 5i$, $-4 + 5i$. In each case, graph the number on the complex plane.

$$z_1 = 4 + 5i \quad \Rightarrow \quad r_1 = \sqrt{41} \quad \theta_1 = \tan^{-1}\left(\frac{5}{4}\right) = 51.3^\circ$$

$$z_2 = -4 - 5i \quad \Rightarrow \quad r_2 = \sqrt{41} \quad \theta_2 = \tan^{-1}\left(\frac{5}{4}\right) + 180^\circ = 231^\circ$$

$$z_3 = 4 - 5i \quad \Rightarrow \quad r_3 = \sqrt{41} \quad \theta_3 = \tan^{-1}\left(-\frac{5}{4}\right) + 360^\circ = 309^\circ$$

$$z_4 = -4 + 5i \quad \Rightarrow \quad r_4 = \sqrt{41} \quad \theta_4 = \tan^{-1}\left(-\frac{5}{4}\right) + 180^\circ = 129^\circ$$

On-line version replaces the 4 and 5 above with random integers between 2 and 9.

Problem 1.15 Express the following numbers in polar form: $3+5i$, $-2-6i$, $5-4i$, $-3+8i$. In each case, graph the number on the complex plane.

$$
\begin{aligned}
z_1 &= 3+5i &\Rightarrow\quad r_1 &= \sqrt{34} \quad \theta_1 = \tan^{-1}\left(\frac{5}{3}\right) = 1.03\,rad\\
z_2 &= -2-6i &\Rightarrow\quad r_2 &= \sqrt{40} \quad \theta_2 = \tan^{-1}\left(\frac{6}{2}\right) + \pi = 4.39\,rad\\
z_3 &= 5-4i &\Rightarrow\quad r_3 &= \sqrt{41} \quad \theta_3 = \tan^{-1}\left(-\frac{4}{5}\right) + 2\pi = 5.61\,rad\\
z_4 &= -3+8i &\Rightarrow\quad r_4 &= \sqrt{73} \quad \theta_4 = \tan^{-1}\left(-\frac{8}{3}\right) + \pi = 1.93\,rad
\end{aligned}
$$

On-line version replaces $\mathrm{Re}\,(z)$ and $\mathrm{Im}\,(z)$ with random integers between 2 and 9 without changing the quadrant for each case above.

Problem 1.16 Express the following in Cartesian form: $(3+5i)/(4-7i)$, $(-3+6i)/(3+9i)$.

$$
z_1 = \frac{(3+5i)}{(4-7i)} = \frac{(3+5i)}{(4-7i)}\frac{(4+7i)}{(4+7i)} = \frac{-23+41i}{65} = -0.354+0.631i
$$
$$
z_2 = \frac{(-3+6i)}{(3+9i)} = \frac{(-3+6i)}{(3+9i)}\frac{(3-9i)}{(3-9i)} = \frac{45+45i}{90} = 2+2i
$$

On-line version randomizes real and imaginary parts of both numerators and denominators.

Problem 1.17 Let $z_1 = 10\,e^{0.5\,i}$ and $z_2 = 20\,e^{-0.8\,i}$. Find the real part, imaginary part, and polar form of
a) $z_1 + z_2$
b) $z_1 - z_2$
c) $z_1\,z_2$
d) z_1/z_2

For (a) and (b), begin by computing the real and imaginary parts, recording an extra digit or so to avoid round-off error.

$$
\begin{aligned}
\mathrm{Re}[z_1] &= 10\cos(0.5) = 8.776 & \mathrm{Im}[z_1] &= 10\sin(0.5) = 4.794\\
\mathrm{Re}[z_2] &= 20\cos(-0.8) = 13.93 & \mathrm{Im}[z_2] &= 20\sin(-0.8) = -14.35
\end{aligned}
$$

Thus:

$$
z_1 + z_2 = 22.7 - 9.56i = 26.6e^{5.88i}
$$

$$z_1 - z_2 = -5.15 + 19.1i = 19.8e^{2.88i}$$

Multiplication and division are easier in polar form:

$$z_1\, z_2 = 200e^{-0.3i} = 200\left[\cos\left(-0.3i\right) + i\sin\left(-0.3i\right)\right] = 191 - 59.1i$$

$$\frac{z_1}{z_2} = 0.5e^{1.3i} = 0.134 + 0.482i$$

On-line version randomizes the magnitude and angle of both z_1 and z_2.

Problem 1.18 For each of the following complex numbers, find the real part and the imaginary part, and express the number in polar form.
a) $z = (4 + 5i)^2$
b) $z = 5(1 + i)e^{i\pi/6}$

$$(4+5i)^2 = (4+5i)\,(4+5i) = -9 + 40i = 41e^{1.79i}$$

$$5\,(1+i)\,e^{i\frac{\pi}{6}} = (5+5i)\,(0.866+0.5i) = 1.83 + 6.83i = 7.07e^{1.31i}$$

On-line version randomizes both subparts.

Problem 1.19 Show explicitly that the following functions are solutions to the spherically symmetric wave equation.

$$(a) \quad \Psi(r,t) = \frac{A}{r}\cos\left(kr \mp \omega t + \varphi\right)$$

$$(b) \quad \Psi(r,t) = \frac{A}{r}e^{i((kr \mp \omega t + \varphi)}$$

In each case, substitute $\Psi(r,t)$ into the differential wave equation.

On-line version uses the complex representation:

$$\Psi\left(r,t\right) = \frac{A}{r}\exp\left[i\left(kr - \omega t + \varphi\right)\right]$$

Answers are entered symbolically, paying close attention to the indicated factorization. *Only the expressions inside the curly braces are entered into the symbolic answer field.* Note the multiplication symbols, which must be entered explicitly.

$$\begin{aligned}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) &= \frac{1}{r^2}\frac{\partial}{\partial r}\left\{-1 + i*k*r\right\}A\exp\left[i\left(kr - \omega t + \varphi\right)\right] \\ &= \frac{1}{r^2}\left\{-k^2*r\right\}A\exp\left[i\left(kr - \omega t + \varphi\right)\right] \\ &= \left\{-k^2\right\}\Psi(r,t)\end{aligned}$$

Similarly,

$$\frac{\partial \Psi}{\partial t} = \{-i * \omega\} \Psi(r,t)$$
$$\frac{\partial^2 \Psi}{\partial t^2} = \left\{-\omega^2\right\} \Psi(r,t)$$

Check for a solution:

$$\frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2} = \left\{\frac{-\omega^2}{v^2}\right\} \Psi(r,t)$$

The spherically symmetric wave equation will be satisfied provided

$$v = \left\{\frac{\omega}{k}\right\}$$

Problem 1.20 Find the equation for a plane electromagnetic wave whose propagation vector $\vec{k}$ is parallel to a line in the x-z plane that is $30°$ measured counter clockwise from the positive x-axis. Assume that $\vec{k}$ lies in the first quadrant of the x-z plane.

Using the complex representation,

$$\Psi(x,y,z,t) = A\exp[k_x x + k_y y + k_z z - \omega t + \varphi]$$

with

$$k_x = k\cos(30°) = 0.866k$$
$$k_y = 0$$
$$k_z = k\sin(30°) = 0.5k$$

Thus

$$\Psi(x,y,z,t) = A\exp[0.866kx + 0.5kz - \omega t + \varphi]$$

On-line version uses a symbolic value for the angle theta. The expressions inside the curly braces are entered into the symbolic answer fields:

$$k_x = \{k * \cos(\theta)\}$$
$$k_y = \{0\}$$
$$k_z = \{k * \sin(\theta)\}$$

Problem 1.21 Determine the direction of propagation of the following harmonic traveling waves:
a) $\Psi(z,t) = A\sin(kz - \omega t)$
b) $\Psi(y,t) = A\cos(\omega t - ky)$

c) $\Psi(x,t) = A\cos(\omega t + kx)$
d) $\Psi(x,t) = A\cos(-\omega t - kx)$

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a) +z
b) +y
c) -x
b) -x

On-line version: correct answers are selected from drop-down lists.

Problem 1.22 Show that the *Gaussian* wave $\Psi(x,t) = Ae^{-a(bx\,-\,ct)^2}$ is a solution to the one-dimensional wave equation. If $a = 5.00$, $b = 10.0$, $c = 100$, with $a(bx - ct)^2$ unitless, determine the wave speed.

Recasting the wavefuntion as a general forward traveling wave gives a velcity of $c/b = 10\,m/s$.

On-line version randomizes a, b and c.

Problem 1.23 Sketch or plot the following wavefunction at times $t = 0$, $t = 0.5\,s$, and $t = 1.0\,s$:

$$\Psi(x,t) = \frac{1.0}{1 + (x + 10t)^2}$$

At $t = 0$, this is a pulse centered at $x = 0$. At $t = 0.5\,s$, the pulse is centered at $x = -5\,m$, and at $t = 1.0\,s$ it is centered at $x = -10\,m$. The sketch or plot should be similar to Figure 1.2, except that the pulse moves in the negative x-direction.

Problem 1.24 A 1-D harmonic traveling wave that travels in the $-y$ direction has amplitude 10 (unitless), wavelength $10.0\,m$, period $2.0\,s$ and initial phase π. Using the complex representation, find an expression for this wave that uses angular frequency and propagation constant.

The angular frequency is

$$\omega = \frac{2\pi}{T} = \pi$$

and the wavenumber is

$$k = \frac{2\pi}{\lambda} = \frac{\pi}{5}$$

The wavefunction is

$$\Psi(y,t) = 10\exp\left[i\left(\frac{\pi}{5}y + \pi t + \pi\right)\right]$$

On-line version uses symbolic input with symbolic values for k and ω. The wavefunction travels in the $-x$ direction and has an initial phase of $\pi/8$. The symoblic answer is entered as the input inside the curly braces:

$$\Psi(y,t) = A\exp\left\{i * \left(k * x + \omega * t + \frac{\pi}{8}\right)\right\}$$

Problem 1.25 Light from a helium-neon laser has a wavelength of $633\,nm$ and a wave speed of $3.00 \times 10^8\ m/s$. Find the frequency, period, angular frequency, and wave number for this light.

$$f = \frac{c}{\lambda} = \frac{3 \times 10^8}{633 \times 10^{-9}} = 4.74 \times 10^{14} Hz$$

$$T = \frac{1}{f} = 2.11 \times 10^{-15} s$$

$$\omega = 2\pi f = 2.98 \times 10^{15} rad/s$$

$$k = \frac{\omega}{c} = 9.92 \times 10^6 m^{-1}$$

On-line version is not randomized.

Problem 1.26 Consider a harmonic wave given by

$$\Psi(x,t) = U(x,y,z)e^{-i\omega t}$$

where $U(x,y,z)$ is called the *complex amplitude*. Show that U satisfies the *Helmholz equation*:

$$\left(\nabla^2 + k^2\right) U(x,y,z) = 0$$

where k is the propagation constant.

Since Ψ is a wave, it satisfies the differential wave equation:

$$\nabla^2 \Psi = \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2}$$

Since U only depends upon space coordinates,

$$\frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2} = \frac{1}{v^2}\left(-\omega^2 U e^{-i\omega t}\right) = -k^2 U e^{-i\omega t} = \left(\nabla^2 U\right) e^{-i\omega t}$$

Cancelling the exponential terms and treating the Laplacian as an operator gives

$$\left(\nabla^2 + k^2\right) U\left(x, y, z\right) = 0$$

Problem 1.27 Show that a complex number divided by its complex conjugate gives a result whose magnitude is one.

Let

$$z = \frac{z_1}{z_1^*}$$

where z and z_1 are complex numbers. The magnitude of z is given by

$$|z| = \sqrt{z\,z^*} = \sqrt{\frac{z_1}{z_1^*}\left(\frac{z_1}{z_1^*}\right)^*} = \sqrt{\frac{z_1}{z_1^*}\frac{z_1^*}{z_1}} = 1$$

Problem 1.28 Show that $z = \frac{\sqrt{2}}{2}(1+i)$ is a square root of i. Find another one.

$$\left[\frac{\sqrt{2}}{2}\left(1+i\right)\right]^2 = \frac{1}{2}\left(2i\right) = i$$

The other root is

$$z = -\frac{\sqrt{2}}{2}(1+i)$$

Problem 1.29 Find the real and imaginary parts of
a) $z = \left(2e^{i\frac{\pi}{4}}\right)^3$
b) $z = \left(2+3i\right)^3$

$$\left(2e^{i\frac{\pi}{4}}\right)^3 = 8e^{i\frac{3\pi}{4}} = 8\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right] = -5.66 + 5.66i$$

$$\left(2+3i\right)^3 = \left(2+3i\right)\left(-3+12i\right) = -42 + 15i$$

On-line version randomizes both problem subparts.

Problem 1.30 Hyperbolic sines and cosines are defined as follows: $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$. Show that $\sin(ix) = i\sinh(x)$ and $\cos(ix) = \cosh(x)$.

$$\sin(ix) = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = \frac{e^{-x} - e^{x}}{2i} = \frac{i}{i}\left(\frac{e^{-x} - e^{x}}{2i}\right) = i\left(\frac{e^{x} - e^{-x}}{2}\right) = i\sinh(x)$$

$$\cos(ix) = \frac{e^{i(ix)} + e^{-i(ix)}}{2} = \frac{e^{-x} + e^{x}}{2} = \cosh(x)$$

Problem 1.31 Find the Taylor series expansions for hyperbolic sine and hyperbolic cosine.

Compute the Taylor series explicitly as in Example 1.3, or use the Taylor series for each exponential term:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + ...$$

Thus

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + ...\right)}{2}$$
$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...$$
$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + ...\right)}{2}$$
$$= x + \frac{x^3}{3!} + \frac{x^5}{5!}...$$

Problem 1.32 Consider a vector $\vec{v}$, and let a be the angle between $\vec{v}$ and the positive x-axis, b be the angle between $\vec{v}$ and the positive y-axis, and c be the angle between $\vec{v}$ and the positive z-axis. Define the *direction cosines* α, β, and γ as follows:

$$\alpha = \cos a = \frac{v_x}{|v|}$$

$$\beta = \cos b = \frac{v_y}{|v|}$$

$$\gamma = \cos c = \frac{v_z}{|v|}$$

a) Show that $\alpha^2 + \beta^2 + \gamma^2 = 1$.
b) Show that the function

$$\Psi(x, y, z, t) = Ae^{i[k(\alpha x + \beta y + \gamma z) \mp \omega t]}$$

is a three-dimensional plane wave that solves the differential wave equation in Cartesian coordinates.

According to the definition given,

$$\alpha^2 + \beta^2 + \gamma^2 = \frac{v_x^2}{v^2} + \frac{v_y^2}{v^2} + \frac{v_z^2}{v^2} = \frac{v^2}{v^2} = 1$$

For part (b), we must substitute into the three-dimensional wave equation in Cartesian coordinates. The x derivates are

$$\frac{\partial \Psi}{\partial x} = ik\alpha\Psi \quad \frac{\partial^2 \Psi}{\partial x^2} = -k^2\alpha^2\Psi$$

with similar expressions for y and z. Thus

$$\nabla^2\Psi = -k^2\left(\alpha^2 + \beta^2 + \gamma^2\right)\Psi = -k^2\Psi$$

Substitution into the wave equation gives

$$\frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2} = \frac{-\omega^2}{v^2}\Psi = -k^2\Psi = \nabla^2\Psi$$

Problem 1.33 The Laplacian in cylindrical coordinates (ρ, φ, z) is given by

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}$$

A *cylindrical wave* has a wavefront that is constant on a cylinder. In other words, it does not depend upon φ or z. Show that

$$\Psi = \frac{A}{\sqrt{\rho}}e^{i(k\rho \mp \omega t)}$$

approximately solves the differential wave equation in clylindrical coordinates for values of ρ that are sufficiently large.

Taking the derivatives in the Laplacian:

$$\frac{\partial\Psi}{\partial\rho} = -\frac{\Psi}{2\rho} + ik\Psi$$

$$\begin{aligned}\frac{\partial^2\Psi}{\partial\rho^2} &= -\frac{1}{2\rho}\left(\frac{\partial\Psi}{\partial\rho}\right) + \frac{\Psi}{2\rho^2} + ik\left(\frac{\partial\Psi}{\partial\rho}\right)\\ &= -\frac{1}{2\rho}\left(-\frac{\Psi}{2\rho} + ik\Psi\right) + \frac{\Psi}{2\rho^2} + ik\left(-\frac{\Psi}{2\rho} + ik\Psi\right)\\ &= \frac{3\Psi}{4\rho^2} - \frac{ik}{\rho}\Psi - k^2\Psi\end{aligned}$$

Thus

$$\begin{aligned}\nabla^2\Psi &= \frac{1}{\rho}\frac{\partial\Psi}{\partial\rho} + \frac{\partial^2\Psi}{\partial\rho^2} \\ &= \frac{1}{\rho}\left(\frac{\Psi}{2\rho} + ik\Psi\right) + \frac{3\Psi}{4\rho^2} - \frac{ik}{\rho}\Psi - k^2\Psi \\ &= \frac{\Psi}{4\rho^2} - k^2\Psi\end{aligned}$$

The wave equation is approximately satisfied provided the first term in the last result can be neglected.

Problem 1.34 Show that the spherically symmetric wave equation can be written as

$$\frac{\partial^2}{\partial r^2}\left[r\Psi\left(r,t\right)\right] = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\left[r\Psi\left(r,t\right)\right]$$

which is a *linear* wave equation in the quantity $r\Psi\left(r,t\right)$. Show that Equations 1.59 and 1.60 are solutions.

Taking the derivatives on the right hand side:

$$\frac{\partial}{\partial r}\left[r\Psi\right] = \Psi + r\frac{\partial\Psi}{\partial r}$$

$$\frac{\partial^2}{\partial r^2}\left[r\Psi\right] = \frac{\partial}{\partial r}\left\{\Psi + r\frac{\partial\Psi}{\partial r}\right\} = 2\frac{\partial\Psi}{\partial r} + r\frac{\partial^2\Psi}{\partial r^2}$$

The left hand side gives

$$\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\left[r\Psi\right] = \frac{r}{v^2}\frac{\partial^2\Psi}{\partial t^2}$$

Thus

$$\frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2} = \frac{1}{r}\left\{2\frac{\partial\Psi}{\partial r} + r\frac{\partial^2\Psi}{\partial r^2}\right\} = \frac{2}{r}\frac{\partial\Psi}{\partial r} + \frac{\partial^2\Psi}{\partial r^2}$$

This the the spherically symmetric wave equation, since

$$\begin{aligned}\nabla^2\Psi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial\Psi}{\partial r}\right] \\ &= \frac{1}{r^2}\left[2r\frac{\partial\Psi}{\partial r} + r^2\frac{\partial^2\Psi}{\partial r^2}\right] \\ &= \frac{2}{r}\frac{\partial\Psi}{\partial r} + \frac{\partial^2\Psi}{\partial r^2}\end{aligned}$$

Problem 1.35 *Photons* are particles of light with energy and momentum given by

$$E = hf$$

$$p = \frac{h}{\lambda}$$

where f is the light frequency, λ is the light wavelength, and h is *Planck's constant*: $h = 6.626 \times 10^{-34}\, J \cdot s$. Show that for photons, $E = cp$.

$$cp = c\frac{h}{\lambda} = hf = E$$
