2 Linear Algebra

Exercises

2.1 (a) The 2×2 and 3×3 discrete derivative matrices can actually be read off from the general form provided there:

$$\mathbb{D}_{2\times 2} = \begin{pmatrix} 0 & \frac{1}{2\Delta x} \\ -\frac{1}{2\Delta x} & 0 \end{pmatrix}, \tag{2.1}$$

$$\mathbb{D}_{3\times 3} = \begin{pmatrix} 0 & \frac{1}{2\Delta x} & 0 \\ -\frac{1}{2\Delta x} & 0 & \frac{1}{2\Delta x} \\ 0 & -\frac{1}{2\Delta x} & 0 \end{pmatrix}. \tag{2.2}$$

(b) To calculate the eigenvalues, we construct the characteristic equation where

$$\det(\mathbb{D}_{2\times 2} - \lambda \mathbb{I}) = 0 = \lambda^2 + \frac{1}{4\Delta x^2}. \tag{2.3}$$

Then, the eigenvalues of the 2×2 derivative matrix are

$$\lambda = \pm \frac{i}{2\Delta x}. \tag{2.4}$$

For the 3×3 derivative matrix, its characteristic equation is

$$\begin{aligned} \det(\mathbb{D}_{3\times 3} - \lambda \mathbb{I}) = 0 = \det \begin{pmatrix} -\lambda & \frac{1}{2\Delta x} & 0 \\ -\frac{1}{2\Delta x} & -\lambda & \frac{1}{2\Delta x} \\ 0 & -\frac{1}{2\Delta x} & -\lambda \end{pmatrix} \\ = (-\lambda)\left(\lambda^2 + \frac{1}{4\Delta x^2}\right) - \frac{\lambda}{4\Delta x^2}. \end{aligned} \tag{2.5}$$

One eigenvalue is clearly 0, and the other eigenvalues satisfy

$$\lambda^2 + \frac{1}{2\Delta x^2} = 0, \tag{2.6}$$

and so the 3×3 derivative matrix has eigenvalues

$$\lambda = 0, \pm \frac{i}{\sqrt{2}\Delta x}. \tag{2.7}$$

All non-zero eigenvalues are exclusively imaginary numbers.

(c) For the 2×2 derivative matrix, the eigenvector equation can be expressed as

$$\begin{pmatrix} 0 & \frac{1}{2\Delta x} \\ -\frac{1}{2\Delta x} & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \pm \frac{i}{2\Delta x} \begin{pmatrix} a \\ b \end{pmatrix}, \tag{2.8}$$

for some numbers a, b. This linear equation requires that

$$b = \pm ia, \tag{2.9}$$

and so the eigenvectors can be expressed as

$$\vec{v}_1 = a \begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad \vec{v}_2 = a \begin{pmatrix} 1 \\ -i \end{pmatrix}. \tag{2.10}$$

To ensure that they are unit normalized, we require that

$$\vec{v}_1^* \cdot \vec{v}_1 = a^2 \, (1 \ \ -i) \begin{pmatrix} 1 \\ i \end{pmatrix} = 2a^2, \tag{2.11}$$

or that $a = 1/\sqrt{2}$ (ignoring a possible overall complex phase). Thus, the normalized eigenvectors are

$$\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad \vec{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}. \tag{2.12}$$

Note that these are mutually orthogonal:

$$\vec{v}_1^* \cdot \vec{v}_2 = \frac{1}{2} (1 \ \ -i) \begin{pmatrix} 1 \\ -i \end{pmatrix} = 1 - 1 = 0. \tag{2.13}$$

For the 3×3 derivative matrix, we will first determine the eigenvector corresponding to 0 eigenvalue, where

$$\begin{pmatrix} 0 & \frac{1}{2\Delta x} & 0 \\ -\frac{1}{2\Delta x} & 0 & \frac{1}{2\Delta x} \\ 0 & -\frac{1}{2\Delta x} & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.14}$$

Performing the matrix multiplication, we find that

$$\begin{pmatrix} b \\ c - a \\ -b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.15}$$

This then enforces that $b = 0$ and $c = a$. That is, the normalized eigenvector with 0 eigenvalue is (again, up to an overall complex phase)

$$\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \tag{2.16}$$

Next, the eigenvectors for the non-zero eigenvalues satisfy

$$\begin{pmatrix} 0 & \frac{1}{2\Delta x} & 0 \\ -\frac{1}{2\Delta x} & 0 & \frac{1}{2\Delta x} \\ 0 & -\frac{1}{2\Delta x} & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \pm \frac{i}{\sqrt{2}\Delta x} \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \tag{2.17}$$

Performing the matrix multiplication, we find

$$\frac{1}{2\Delta x}\begin{pmatrix} b \\ c-a \\ -b \end{pmatrix} = \pm\frac{i}{\sqrt{2}\Delta x}\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \tag{2.18}$$

or that

$$b = \pm\sqrt{2}i\,a\,, \qquad c = \pm\frac{i}{\sqrt{2}}b = -a\,. \tag{2.19}$$

Then, the other two eigenvectors are

$$\vec{v}_2 = a\begin{pmatrix} 1 \\ \sqrt{2}i \\ -1 \end{pmatrix}, \qquad \vec{v}_3 = a\begin{pmatrix} 1 \\ -\sqrt{2}i \\ -1 \end{pmatrix}. \tag{2.20}$$

The value of a can be determined by demanding that they are normalized:

$$\vec{v}_2^* \cdot \vec{v}_2 = a^2\begin{pmatrix} 1 & -\sqrt{2}i & -1 \end{pmatrix}\begin{pmatrix} 1 \\ \sqrt{2}i \\ -1 \end{pmatrix} = 4a^2\,, \tag{2.21}$$

or that $a = 1/2$. That is,

$$\vec{v}_2 = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2}i \\ -1 \end{pmatrix}, \qquad \vec{v}_3 = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2}i \\ -1 \end{pmatrix}. \tag{2.22}$$

All three eigenvectors, $\vec{v}_1, \vec{v}_2, \vec{v}_3$, are mutually orthogonal. For example,

$$\vec{v}_2^* \cdot \vec{v}_3 = \frac{1}{4}\begin{pmatrix} 1 & -\sqrt{2}i & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -\sqrt{2}i \\ -1 \end{pmatrix} = \frac{1-2+1}{4} = 0\,. \tag{2.23}$$

(d) To determine how the exponentiated matrix

$$\mathbb{M} = e^{\Delta x \mathbb{D}} = \sum_{n=0}^{\infty} \frac{\Delta x^n}{n!}\mathbb{D}^n\,, \tag{2.24}$$

acts on an eigenvector $\vec{v}$ with eigenvalue λ, let's use the Taylor expanded form. Note that

$$\mathbb{D}^n \vec{v} = \lambda^n \vec{v}\,, \tag{2.25}$$

and so

$$\mathbb{M}\vec{v} = \sum_{n=0}^{\infty} \frac{\Delta x^n}{n!}\mathbb{D}^n \vec{v} = \sum_{n=0}^{\infty} \frac{(\lambda\,\Delta x)^n}{n!}\vec{v} = e^{\lambda \Delta x}\vec{v}\,. \tag{2.26}$$

We can then just plug in the appropriate eigenvalues and eigenvectors.

(e) Now, we are asked to determine the matrix form of the exponentiated 2×2 and 3×3 derivative matrices. Let's start with the 2×2 matrix and note that we can write

$$\mathbb{M}_{2\times 2} = e^{\Delta x \mathbb{D}_{2\times 2}} = \sum_{n=0}^{\infty} \frac{\Delta x^n}{n!} \begin{pmatrix} 0 & \frac{1}{2\Delta x} \\ -\frac{1}{2\Delta x} & 0 \end{pmatrix}^n = \sum_{n=0}^{\infty} \frac{1}{2^n n!} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^n . \tag{2.27}$$

So, the problem is reduced to establishing properties of the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Note that the first few powers of the matrix are:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I}, \tag{2.28}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.29}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\mathbb{I}. \tag{2.30}$$

This pattern continues, and it can be compactly expressed as

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{2n} = i^{2n}\mathbb{I}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{2n+1} = (-i)i^{2n+1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.31}$$

for $n = 0,1,2,\dots$. Then, the 2×2 exponentiated matrix can be expressed as

$$\begin{aligned} \mathbb{M}_{2\times 2} &= \mathbb{I}\sum_{n=0}^{\infty} \frac{i^{2n}}{2^{2n}(2n)!} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sum_{n=0}^{\infty} \frac{i^{2n}}{2^{2n+1}(2n+1)!} \\ &= \cos\frac{1}{2}\,\mathbb{I} + \sin\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \cos\frac{1}{2} & \sin\frac{1}{2} \\ -\sin\frac{1}{2} & \cos\frac{1}{2} \end{pmatrix}. \end{aligned} \tag{2.32}$$

A similar exercise can be repeated for the 3×3 derivative matrix. We want to evaluate the sum

$$\begin{aligned} \mathbb{M}_{3\times 3} = e^{\Delta x \mathbb{D}_{3\times 3}} &= \sum_{n=0}^{\infty} \frac{\Delta x^n}{n!} \begin{pmatrix} 0 & \frac{1}{2\Delta x} & 0 \\ -\frac{1}{2\Delta x} & 0 & \frac{1}{2\Delta x} \\ 0 & -\frac{1}{2\Delta x} & 0 \end{pmatrix}^n \\ &= \sum_{n=0}^{\infty} \frac{1}{2^n n!} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}^n . \end{aligned} \tag{2.33}$$

Now, note that even powers of the matrix take the form

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}^0 = \mathbb{I}, \tag{2.34}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} - 2\mathbb{I}, \tag{2.35}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}^4 = 2\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} + 4\mathbb{I}, \tag{2.36}$$

and so the general result takes the form

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}^{2n} = (-2)^{n-1}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} + (-2)^n\mathbb{I}, \tag{2.37}$$

for $n = 1, 2, \ldots$. Products of odd powers of the matrix take the form

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}^1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \tag{2.38}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}^3 = -2\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \tag{2.39}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}^5 = 4\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{2.40}$$

The general form is then

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}^{2n+1} = (-2)^n\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \tag{2.41}$$

for $n = 0, 1, 2, \ldots$. Putting these results together, the 3×3 exponentiated matrix is

$$\begin{aligned} \mathbb{M}_{3\times3} &= \mathbb{I} + \sum_{n=1}^{\infty} \frac{(-2)^n}{2^{2n}(2n)!}\left(-\frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \mathbb{I}\right) \\ &\quad + \sum_{n=0}^{\infty} \frac{(-2)^n}{2^{2n+1}(2n+1)!}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{2}^{2n}(2n)!} \\ &\quad + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{2}^{2n+1}(2n+1)!} \end{aligned} \tag{2.42}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \frac{\cos\frac{1}{\sqrt{2}}}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} + \frac{\sin\frac{1}{\sqrt{2}}}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

2.2 If we instead defined the derivative matrix through the standard asymmetric difference, the derivative matrix would take the form

$$\mathbb{D} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \cdots \\ \cdots & -\frac{1}{\Delta x} & \frac{1}{\Delta x} & 0 & \cdots \\ \cdots & 0 & -\frac{1}{\Delta x} & \frac{1}{\Delta x} & \cdots \\ \cdots & 0 & 0 & -\frac{1}{\Delta x} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.43}$$

Such a matrix has no non-zero entries below the diagonal and so its characteristic equation is rather trivial, for any number of grid points. Note that

$$\det(\mathbb{D}_{n\times n} - \lambda\mathbb{I}) = \det\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \cdots \\ \cdots & -\frac{1}{\Delta x}-\lambda & \frac{1}{\Delta x} & 0 & \cdots \\ \cdots & 0 & -\frac{1}{\Delta x}-\lambda & \frac{1}{\Delta x} & \cdots \\ \cdots & 0 & 0 & -\frac{1}{\Delta x}-\lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= 0 = \left(-\frac{1}{\Delta x} - \lambda\right)^n. \tag{2.44}$$

Thus, there is but a single eigenvalue, $\lambda = -1/\Delta x$.

2.3 (a) We are asked to express the quadratic polynomial $p(x)$ as a linear combination of Legendre polynomials. So, we have

$$p(x) = ax^2 + bx + c = d_0P_0(x) + d_1P_1(x) + d_2P_2(x) \tag{2.45}$$
$$= d_0\frac{1}{\sqrt{2}} + d_1\sqrt{\frac{3}{2}}x + d_2\left(\sqrt{\frac{5}{8}}(3x^2-1)\right),$$

for some coefficients d_0, d_1, d_2. First, matching the coefficient of x^2, we must have

$$3\sqrt{\frac{5}{8}}d_2 = a, \tag{2.46}$$

or that

$$d_2 = \frac{1}{3}\sqrt{\frac{8}{5}}a. \tag{2.47}$$

Next, matching coefficients of x, we have

$$d_1\sqrt{\frac{3}{2}} = b, \tag{2.48}$$

or that

$$d_1 = \sqrt{\frac{2}{3}}\, b\,. \tag{2.49}$$

Finally, matching coefficients of x^0, we have

$$d_0 \frac{1}{\sqrt{2}} - d_2 \sqrt{\frac{5}{8}} = d_0 \frac{1}{\sqrt{2}} - \frac{a}{3} = c\,, \tag{2.50}$$

or that

$$d_0 = \sqrt{2}c + \frac{\sqrt{2}}{3} a\,. \tag{2.51}$$

Then, we can express the polynomial as the linear combination

$$p(x) = \left(\sqrt{2}c + \frac{\sqrt{2}}{3} a\right) P_0(x) + \sqrt{\frac{2}{3}}\, b P_1(x) + \frac{1}{3}\sqrt{\frac{8}{5}}\, a P_2(x)\,, \tag{2.52}$$

or as the vector in the space of Legendre polnomials

$$p(x) = \left(\begin{array}{c} \sqrt{2}c + \frac{\sqrt{2}}{3} a \\ \sqrt{\frac{2}{3}}\, b \\ \frac{1}{3}\sqrt{\frac{8}{5}}\, a \end{array}\right). \tag{2.53}$$

(b) Let's now act on this vector with the derivative matrix we constructed:

$$\frac{d}{dx} p(x) = \left(\begin{array}{ccc} 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{15} \\ 0 & 0 & 0 \end{array}\right) \left(\begin{array}{c} \sqrt{2}c + \frac{\sqrt{2}}{3} a \\ \sqrt{\frac{2}{3}}\, b \\ \frac{1}{3}\sqrt{\frac{8}{5}}\, a \end{array}\right) = \left(\begin{array}{c} \sqrt{2}b \\ \sqrt{\frac{8}{3}}a \\ 0 \end{array}\right). \tag{2.54}$$

Then, re-interpreting this as a polynomial, we have that

$$\frac{d}{dx} p(x) = \sqrt{2} b P_0(x) + \sqrt{\frac{8}{3}} a P_1(x) = b + 2ax\,, \tag{2.55}$$

which is indeed the derivative of $p(x) = ax^2 + bx + c$.

(c) Let's first construct the second derivative matrix through squaring the first derivative matrix:

$$\frac{d^2}{dx^2} = \frac{d}{dx}\frac{d}{dx} = \left(\begin{array}{ccc} 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{15} \\ 0 & 0 & 0 \end{array}\right) \left(\begin{array}{ccc} 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{15} \\ 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{ccc} 0 & 0 & 3\sqrt{5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right). \tag{2.56}$$

By contrast, the explicit matrix element in the ij position would be

$$\left(\frac{d^2}{dx^2}\right)_{ij} = \int_{-1}^{1} dx\, P_{i-1}(x) \frac{d^2}{dx^2} P_{j-1}(x)\,. \tag{2.57}$$

For the second derivative to be non-zero, it must act on $P_2(x)$, and so $j = 3$. Further, by orthogonality of the Legendre polynomials, the only non-zero value is $i = 1$. Then, the one non-zero element in the matrix is

$$\left(\frac{d^2}{dx^2}\right)_{13} = \int_{-1}^{1} dx\, P_0(x) \frac{d^2}{dx^2} P_2(x) = \frac{1}{\sqrt{2}} \int_{-1}^{1} dx\, 3\sqrt{\frac{5}{2}} = 3\sqrt{5}\,, \tag{2.58}$$

which agrees exactly with just squaring the derivative matrix.

(d) To calculate the exponential of the derivative matrix, we need all of its powers. We have already calculated the first and second derivative matrices, but what about higher powers? With only the first three Legendre polynomials, it is easy to see that the third and higher derivative matrices are all 0:

$$\frac{d^3}{dx^3} = \frac{d}{dx}\frac{d^2}{dx^2} = \begin{pmatrix} 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{15} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 3\sqrt{5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.59}$$

Thus, the Taylor expansion of the exponentiated derivative matrix terminates after a few terms:

$$\mathbb{M} = e^{\Delta x \frac{d}{dx}} = \mathbb{I} + \Delta x \begin{pmatrix} 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{15} \\ 0 & 0 & 0 \end{pmatrix} + \frac{\Delta x^2}{2} \begin{pmatrix} 0 & 0 & 3\sqrt{5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.60}$$

The action of this matrix on the polynomial as expressed as a vector in Legendre polynomial space is

$$\begin{aligned} \mathbb{M} \begin{pmatrix} \sqrt{2}c + \frac{\sqrt{2}}{3}a \\ \sqrt{\frac{2}{3}}\,b \\ \frac{1}{3}\sqrt{\frac{8}{5}}\,a \end{pmatrix} &= \begin{pmatrix} \sqrt{2}c + \frac{\sqrt{2}}{3}a \\ \sqrt{\frac{2}{3}}\,b \\ \frac{1}{3}\sqrt{\frac{8}{5}}\,a \end{pmatrix} + \Delta x \begin{pmatrix} \sqrt{2}b \\ \sqrt{\frac{8}{3}}a \\ 0 \end{pmatrix} + \Delta x^2 \begin{pmatrix} \sqrt{8}a \\ 0 \\ 0 \end{pmatrix} \\ &= ax^2 + bx + c + (b + 2a)\Delta x + 2a\Delta x^2 \\ &= a(x + \Delta x)^2 + b(x + \Delta x) + c\,, \end{aligned} \tag{2.61}$$

which is indeed a translation of the polynomial, as expected.

2.4 (a) To determine if integration is linear, we need to verify the two properties. First, for two functions $f(x)$ and $g(x)$, anti-differentiation acts on their sum as:

$$\int dx\, (f(x) + g(x)) = F(x) + G(x) + c\,, \tag{2.62}$$

where we have

$$\frac{d}{dx}F(x) = f(x)\,, \qquad \frac{d}{dx}G(x) = g(x)\,, \tag{2.63}$$

and c is an arbitrary constant. Linearity requires that this equal the sum of their anti-derivatives:

$$\int dx\, f(x) + \int dx\, g(x) = F(x) + G(x) + 2c\,. \tag{2.64}$$

Note that each integral picks up a constant c. So, the only way that anti-differentiation can be linear is if the integration constant $c = 0$. Note also that integration, i.e., the area under a curve, is independent of the integration constant.

Next, we must also demand that multiplication by a constant is simple for linearity. That is, for some constant c, we must have that

$$\int dx\, c\, f(x) = c\, F(x)\,, \tag{2.65}$$

which is indeed true by the Leibniz product rule.

(b) For two vectors $\vec{f}, \vec{g}$ related by the action of the differentiation matrix,

$$\mathbb{D}\vec{f} = \vec{g}\,, \tag{2.66}$$

we would like to define the anti-differentiation operator $\mathbb{A}$ as

$$\vec{f} = \mathbb{D}^{-1}\vec{g} = \mathbb{A}\vec{g}\,. \tag{2.67}$$

However, this clearly is only well-defined if the derivative operator is invertible; or, that it has no 0 eigenvalues. For the 3×3 derivative matrix, we found that it had a 0 eigenvalue, and this property holds for higher dimensional differentiation matrices. Thus, some other restrictions must be imposed for anti-differentiation to be well-defined as a matrix operator.

2.5 (a) As a differential equation, the eigenvalue equation for the operator $\hat{S}$ is

$$-ix\frac{d}{dx} f_\lambda(x) = \lambda\, f_\lambda(x)\,, \tag{2.68}$$

for some eigenvalue λ and eigenfunction f_λ. This can be rearranged into

$$\frac{df_\lambda}{f_\lambda} = i\lambda\frac{dx}{x}\,, \tag{2.69}$$

and has a solution

$$f_\lambda(x) = c\, x^{i\lambda} = c\, e^{i\lambda \log x}\,, \tag{2.70}$$

for some constant c. Note that the logarithm only makes sense if $x \geq 0$, so we restrict the domain of the operator $\hat{S}$ to $x \in [0,\infty)$. (This can be relaxed, but you have to define what you mean by logarithm of a negative number carefully.) For the eigenfunctions to be bounded, we must require that λ is real-valued.

(b) Integration of the product of two eigenfunctions with eigenvalues $\lambda_1 \neq \lambda_2$ yields

$$\int_0^\infty dx\, f_{\lambda_1}(x)^* f_{\lambda_2}(x) = \int_0^\infty dx\, x^{-i(\lambda_1-\lambda_2)} = \int_0^\infty dx\, e^{-i(\lambda_1-\lambda_2)\log x}\,, \tag{2.71}$$

where we restrict to the domain $x \in [0,\infty)$, as mentioned above. Now, let's change variables to $y = \log x$, and now $y \in (-\infty,\infty)$, and the integral becomes

$$\int_0^\infty dx\, e^{-i(\lambda_1-\lambda_2)\log x} = \int_{-\infty}^\infty dy\, e^y\, e^{-i(\lambda_1-\lambda_2)y}\,. \tag{2.72}$$

Note the extra factor e^y in the integrand; this is the Jacobian of the change of variables $x = e^y$, and so $dx = e^y\, dy$. If this Jacobian were not there, then the integral would be exactly like we are familiar with from Fourier transforms. However, with it there, this integral is not defined. We will fix it in later chapters.

(c) A function $g(x)$ with a Taylor expansion about 0 can be expressed as

$$g(x) = \sum_{n=0}^{\infty} a_n x^n\,, \tag{2.73}$$

for some coefficients a_n. If we act $\hat{S}$ on this function, we find

$$\hat{S}g(x) = -i\sum_{n=0}^{\infty} a_n x\frac{d}{dx}x^n = -i\sum_{n=0}^{\infty} n a_n x^n\,. \tag{2.74}$$

Therefore, if m powers of $\hat{S}$ act on $g(x)$, it returns

$$\hat{S}^m g(x) = \sum_{n=0}^{\infty} a_n \left(-ix\frac{d}{dx}\right)^m x^n = \sum_{n=0}^{\infty}(-in)^m a_n x^n\,. \tag{2.75}$$

Now, the action of the exponentiated operator on $g(x)$ is

$$\begin{aligned} e^{i\alpha\hat{S}}g(x) &= \sum_{m=0}^{\infty}\frac{(i\alpha)^m}{m!}\hat{S}^m g(x) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(i\alpha)^m}{m!}(-in)^m a_n x^n \\ &= \sum_{n=0}^{\infty} a_n x^n \sum_{m=0}^{\infty}\frac{(n\alpha)^m}{m!} = \sum_{n=0}^{\infty} e^{n\alpha} a_n x^n = \sum_{n=0}^{\infty} a_n \left(e^{\alpha}x\right)^n \\ &= g\left(e^{\alpha}x\right)\,. \end{aligned} \tag{2.76}$$

That is, this operator rescales the coordinate x.

2.6 (a) For the matrix $\mathbb{M}$, its characteristic equation is

$$\begin{aligned} \det(\mathbb{M} - \lambda\mathbb{I}) &= \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - (a+d)\lambda + (ad-bc) = 0\,. \end{aligned} \tag{2.77}$$

We can write this in a very nice way in terms of the trace and the determinant of $\mathbb{M}$:

$$\det(\mathbb{M} - \lambda\mathbb{I}) = \lambda^2 - (\operatorname{tr}\mathbb{M})\,\lambda + \det\mathbb{M} = 0\,. \tag{2.78}$$

(b) Solving for the eigenvalues, we find

$$\lambda = \frac{\operatorname{tr}\mathbb{M} \pm \sqrt{(\operatorname{tr}\mathbb{M})^2 - 4\det\mathbb{M}}}{2}\,. \tag{2.79}$$

So, the eigenvalues are real iff $(\operatorname{tr}\mathbb{M})^2 \geq 4\det\mathbb{M}$, or, in terms of the matrix elements,

$$(a+d)^2 \geq 4(ad-bc)\,. \tag{2.80}$$