Chapter 1. Introduction to Dynamic Systems

Topics

Representing forces and motions in time and frequency formats; harmonic motion; phasors; (Q1-5)
Dynamic properties and idealisation of structures – mass, stiffness, damping; (Q6-8)
Dynamic loads – impulsive, harmonic, periodic, random (Q9-10)

Problems

1. A mass on a spring is displaced by 10 mm and then released. In 10 s it completes four cycles of harmonic motion. Calculate a) the angular velocity of the equivalent circular motion, b) the maximum velocity of the mass, and c) its maximum acceleration.

2. A harmonic vibration has initial displacement 0.05 m, initial velocity 1.0 m/s and frequency 5.0 Hz. Write expressions for the displacement x, a) as the sum of a sine and a cosine term, b) as a sine term using a phase angle, and c) in complex exponential form.

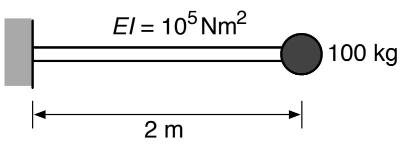
3. Find the magnitude of the sum of the two vectors $4e^{i\pi/12}$ and $5e^{i\pi/3}$, and the angle between the first vector and the resultant.

4. Show that multiplying the vector $z = e^{i\omega t}$ by i has the effect of introducing a phase shift of 90°.

5. Two people exercising in a gym produce vertical forces on the floor which vary roughly sinusoidally with time. In both cases the force has amplitude 250 N and frequency 2.5 Hz, but their peaks are separated in time by 0.05 s. Find an expression for the resultant of the two forces.

6. Each of the systems shown below can be represented as a single mass-spring system for dynamic analysis. Find the appropriate stiffness and mass in each case. The masses of any beams or springs are neglible compared to the lumped masses shown.

a)



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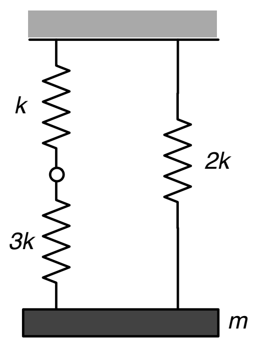

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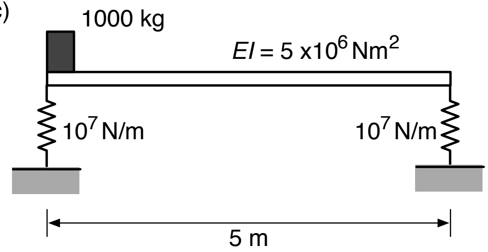

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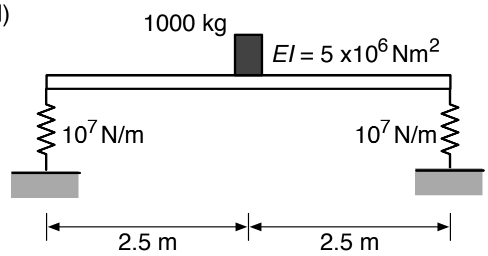
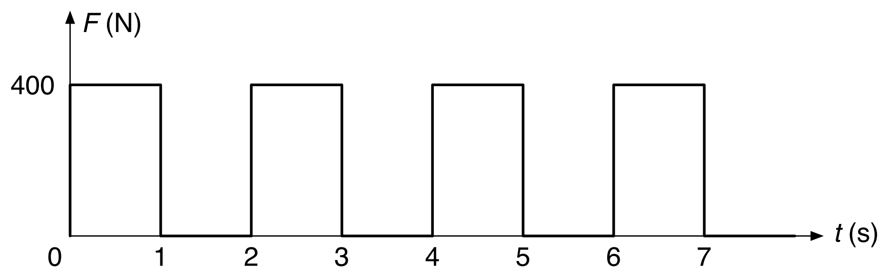


7. For each of the structures in Q6, calculate value of the viscous damping coefficient required to give a damping ratio $\xi = 0.05$.

8. A viscous damper with coefficient $c = 480$ Ns/m is subjected to five cycles of harmonic motion of amplitude 50 mm and frequency 2.0 Hz. Calculate the maximum force in the damper and the energy dissipated. Where does this energy go?

9. A washing machine contains a mass of 1 kg of wet clothes. To dry them, the drum of machine spins at 1000 revs per minute about a horizontal axis. Assuming the clothes are initially formed into a small lump pressed against the side of the drum, of radius 150 mm, estimate the time-varying vertical force induced by the spinning mass. (Recall that a mass undergoing circular motion is accelerated towards the centre of rotation by a centripetal force equal to mass × radius × angular velocity squared.)

10. Express the periodic force below as a Fourier series.



Solutions

1. A mass on a spring is displaced by 10 mm and then released. In 10 s it completes four cycles of harmonic motion. Calculate a) the angular velocity of the equivalent circular motion, b) the maximum velocity of the mass, and c) its maximum acceleration.

   a) Period $T = 10/4 = 2.5$ s. Eq 1.5: Equivalent angular velocity $\omega = 2\pi/T = 2.51$ rad/s

   b) Displacement amplitude $X = 0.01$ m. Eq 1.8: $v(\max) = \omega X = 0.025$ m/s

   c) $a(\max) = \omega^2 X = 0.063$ m/s^2

2. A harmonic vibration has initial displacement 0.05 m, initial velocity 1.0 m/s and frequency 5.0 Hz. Write expressions for the displacement *x,* a) as the sum of a sine and a cosine term, b) as a sine term using a phase angle, and c) in complex exponential form.

   a) Write as $x = A\sin 2\pi ft + B\cos 2\pi ft$ where $f = 5$ Hz

   Differentiate w.r.t. time: $\dot{x} = 2\pi f(A\cos 2\pi ft - B\sin 2\pi ft)$

   Since sin(0) = 0 and cos(0) = 1, the initial conditions give:

   $x(0) = 0.05 = B$;

   $\dot{x}(0) = 1 = 2\pi fA \quad \rightarrow \quad A = 1/(2\pi.5) = 0.00318$

   Hence: $x = 0.0318\sin 10\pi t + 0.05\cos 10\pi t$ (in units of m)

   b) Using the trig. identity for the sine of the sum of two angles, we can write:

   $x = C\sin(2\pi ft + \phi) = C(\cos\phi \sin 2\pi ft + \sin\phi \cos 2\pi ft)$

   Comparing with the solution in a) this implies $C\cos\phi = 0.0318, \quad C\sin\phi = 0.05$

   Therefore $C = (0.0318^2 + 0.05^2)^{1/2} = 0.0593, \quad \tan\phi = (0.05/0.0318) \quad \rightarrow \quad \phi = 1.00$ rads

   Hence $x = 0.0593\sin(10\pi t + 1.0)$

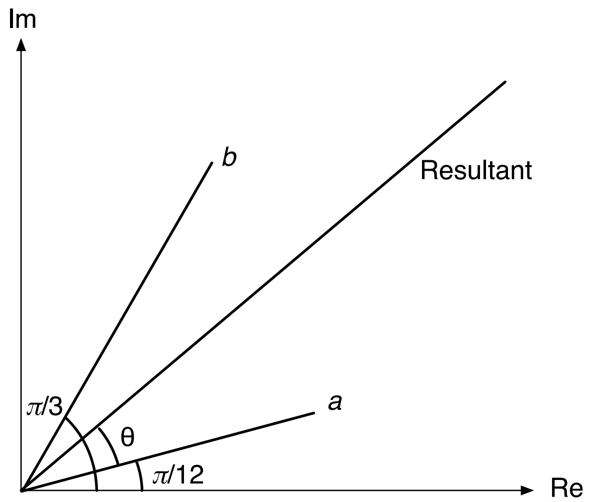
   c) Using Eq 1.14: $x = X\sin(\omega t + \phi) = -\dfrac{iX}{2}(\mathrm{e}^{i(\omega t+\phi)} - \mathrm{e}^{-i(\omega t+\phi)})$

   Comparing with the solution in b): $X = 0.0593$, $\omega = 10\pi$, $\phi = 1.0$, hence:

   $x = -0.0296i(\mathrm{e}^{i(10\pi t+1.0)} - \mathrm{e}^{-i(10\pi t+1.0)})$

3. Find the magnitude of the sum of the two vectors $4e^{i\pi/12}$ and $5e^{i\pi/3}$, and the angle between the first vector and the resultant.

Refer to the Argand diagram below, where $a = 4e^{i\pi/12}$ and $b = 5e^{i\pi/3}$.



Summing the real and imaginary components:

$$\text{Re} = 4\cos\frac{\pi}{12} + 5\cos\frac{\pi}{3} = 6.36, \quad \text{Im} = 4\sin\frac{\pi}{12} + 5\sin\frac{\pi}{3} = 5.37$$

Resultant = $(\text{Re}^2 + \text{Im}^2)^{1/2} = 8.32$

$\tan(\theta + \pi/12) = \text{Im}/\text{Re} = 0.843 \rightarrow \theta = 0.44$ rads

4. Show that multiplying the vector $z = \mathrm{e}^{i\omega t}$ by i has the effect of introducing a phase shift of 90°.

Use Euler's equation, Eq 1.11, and consider the vector z shifted by a phase angle ϕ:

$$\begin{aligned}\mathrm{e}^{i(\omega t+\phi)} &= \cos(\omega t + \phi) + i\sin(\omega t + \phi) \\ &= \cos\omega t\cos\phi - \sin\omega t\sin\phi + i[\sin\omega t\cos\phi + \cos\omega t\sin\phi]\end{aligned}$$

If $\phi = 90°$ then $\sin\phi = 1$, $\cos\phi = 0$, giving:

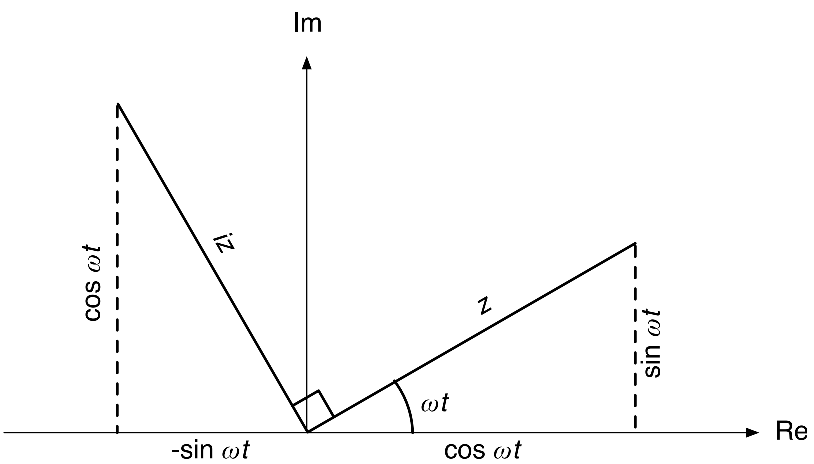
$$\mathrm{e}^{i(\omega t+90°)} = -\sin\omega t + i\cos\omega t = i[\cos\omega t + i\sin\omega t] = i\mathrm{e}^{i\omega t} = iz$$

Alternative solution using Argand diagram:

Euler's equation: $z = \mathrm{e}^{i\omega t} = \cos\omega t + i\sin\omega t$

Multiply by i: $iz = i\cos\omega t + i^2\sin\omega t = -\sin\omega t + i\cos\omega t$

From the plot of these vectors on the Argand diagram, it is clear they are 90° apart.



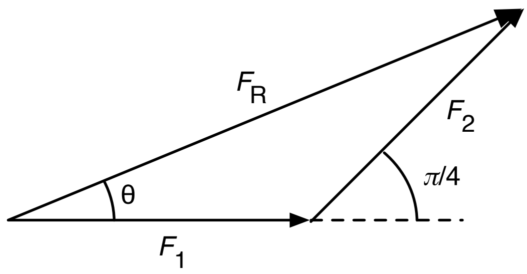
5. Two people exercising in a gym produce vertical forces on the floor which vary roughly sinusoidally with time. In both cases the force has amplitude 250 N and frequency 2.5 Hz, but their peaks are separated in time by 0.05 s. Find an expression for the resultant of the two forces.

The period of the motion is $T = 1/f = 0.4$ s

The two forces are separated by a phase angle of $2\pi.(0.05/0.4) = \pi/4$ rads = 45°.

Therefore we can write: $F_1 = Ae^{i.2\pi ft}$, $F_2 = Ae^{i.(2\pi ft+\pi/4)} = Ae^{i\pi/4}e^{i.2\pi ft}$

where A = 250 N, f = 2.5 Hz. The two phasors have magnitude A and are separated by a phase angle $\pi/4$. Performing vector addition of the phasors:



$$F_R = A[(1+1/\sqrt{2})^2 + (1/\sqrt{2})^2]^{1/2} = 1.85A = 462 \text{ N}$$

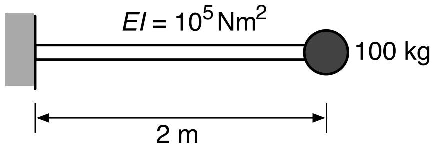
$$\theta = \tan^{-1}\left(\frac{1/\sqrt{2}}{1+1/\sqrt{2}}\right) = \pi/8 \text{ rads} = 22.5°$$

Hence the resultant can be written $F_R = 462e^{i.(2\pi ft+\pi/8)} = 462e^{i\pi.(5t+0.125)}$

where the phase angle is relative to F_1.

6. Each of the systems shown below can be represented as a single mass-spring system for dynamic analysis. Find the appropriate stiffness and mass in each case. The masses of any beams or springs are neglible compared to the lumped masses shown.

a)



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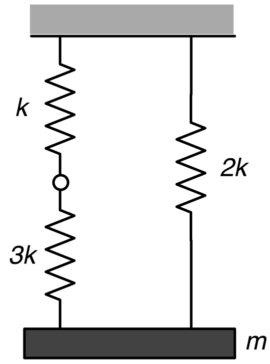

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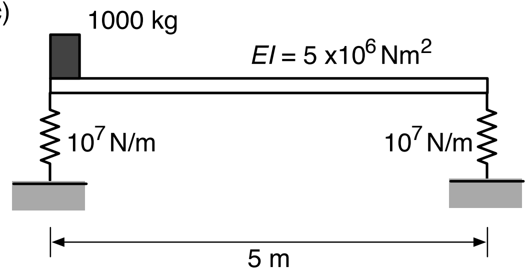

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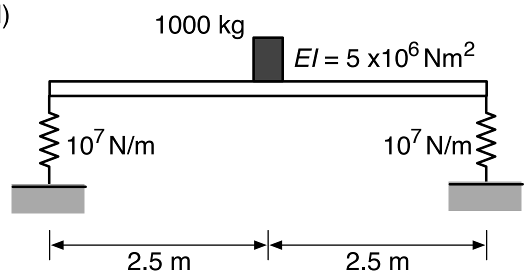


a) Cantilever stiffness $= \dfrac{3EI}{L^3} = \dfrac{3\times10^5}{2^3} = 37{,}500\text{ N/m}$. Therefore k = 37.5 kN/m, m = 100 kg.

b) For the two springs in series, let total extension due to a force F be $e = e_1 + e_2$.

$$e = \frac{F}{k} + \frac{F}{3k} = \frac{4F}{3k} \quad\rightarrow\quad k_{eq} = \frac{F}{e} = \frac{3k}{k}$$

This then acts in parallel with the $2k$ spring, giving a total stiffness of $2.75k$, mass m.

c) With mass directly over left-hand support there is no deformation of beam or right-hand support. Therefore we have simply $k = 10^7$ N/m, m = 1000 kg.

d) With mass at midspan, flexural stiffness of beam is

$$k_B = \frac{48EI}{L^3} = \frac{48\times5\times10^6}{5^3} = 1.92\times10^6\text{ N/m}$$

This acts in series with the support stiffness of $k_S = 2\times10^7$ N/m, giving:

$$\frac{1}{k} = \frac{1}{k_B} + \frac{1}{k_S} \quad\rightarrow\quad k = 1.75\times10^6\text{ N/m},\ m = 1000\text{ kg}$$

7. For each of the structures in Q6, calculate value of the viscous damping coefficient required to give a damping ratio $\xi = 0.05$.

The calculation takes the same form in each case. From Eq. 1.23, $c = \xi.2\sqrt{km} = 0.1\sqrt{km}$, hence:

a) $k = 375{,}000$ N/m, $m = 100$ kg $\rightarrow$ $c = 612.4$ Ns/m

b) $k = 2.75k$, $m = m \rightarrow$ $c = 0.166\sqrt{km}$

c) $k = 10^7$ N/m, $m = 1000$ kg $\rightarrow$ $c = 10{,}000$ Ns/m

d) $k = 1.75 \times 10^6$ N/m, $m = 1000$ kg $\rightarrow$ $c = 4183$ Ns/m

8. A viscous damper with coefficient $c = 480$ Ns/m is subjected to five cycles of harmonic motion of amplitude 50 mm and frequency 2.0 Hz. Calculate the maximum force in the damper and the energy dissipated. Where does this energy go?

The harmonic displacement (in m) can be expressed as $x = 0.05\sin(2\pi.2t)$

Differentiating to find the velocity (m/s): $\dot{x} = 4\pi.0.05\cos(4\pi t) = 0.2\pi\cos(4\pi t)$

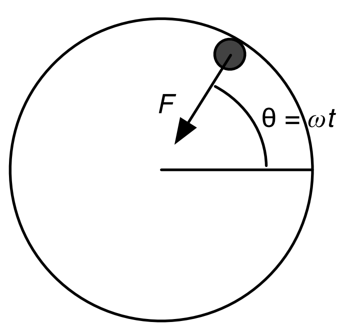
Maximum damper force: $F_D(\max) = c\dot{x}(\max) = 480 \times 0.2\pi = 301.6$ N

Using Eq 1.21: work done $= 5\pi c\omega X^2 = 5 \times \pi \times 480 \times 4\pi \times 0.05^2 = 237$ J

A viscous damper usually works by squeezing a viscous fluid through one or more orifices - the energy goes into heating of the fluid.

9. A washing machine contains a mass of 1 kg of wet clothes. To dry them, the drum of machine spins at 1000 revs per minute about a horizontal axis. Assuming the clothes are initially formed into a small lump pressed against the side of the drum, of radius 150 mm, estimate the time-varying vertical force induced by the spinning mass. (Recall that a mass undergoing circular motion is accelerated towards the centre of rotation by a centripetal force equal to mass $\times$ radius $\times$ angular velocity squared.)

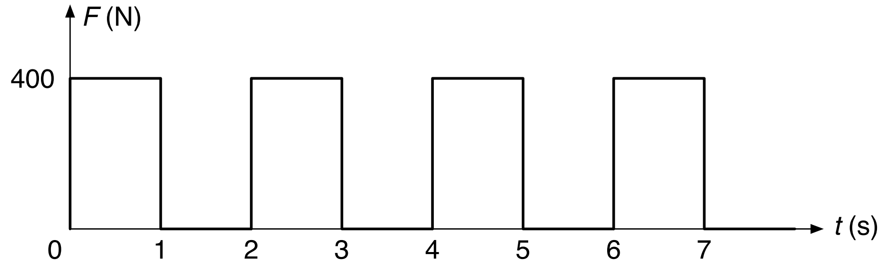
Angular velocity $\omega = \dfrac{1000 \times 2\pi}{60} = 104.7$ rad/s



Centripetal force $F = mr\omega^2 = 1 \times 0.15 \times 104.7^2 = 1645$ N

Vertical force $F_v = F\sin\omega t = 1645\sin(104.7t)$

10. Express the periodic force below as a Fourier series.



Treat as the sum of a constant force of 200 N and a square wave between –200 and +200 N.

This is an odd function so expressed as a Fourier sine series:

$$F = 200 + \sum_{n=1}^{\infty} B_n \sin(n\omega_0 t) \text{ where } \omega_0 = 2\pi/T = 2\pi/2 = \pi \text{ rad/s, and}$$

$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} F(t)\sin(n\omega_0 t)\,\mathrm{d}t = \frac{2}{2}\left(\int_{-1}^{0} -200.\sin(n\pi t)\,\mathrm{d}t + \int_{0}^{1} +200.\sin(n\pi t)\,\mathrm{d}t\right)$$

$$= \frac{200}{n\pi}\left(\left[\cos(n\pi t)\right]_{-1}^{0} + \left[-\cos(n\pi t)\right]_{0}^{1}\right) = \frac{400}{n\pi}\left(1-(-1)^n\right)$$

Terms up to $n = 9$ give:

$$F = 200 + \frac{800}{\pi}\left[\sin(\pi t) + \frac{1}{3}\sin(3\pi t) + \frac{1}{5}\sin(5\pi t) + \frac{1}{7}\sin(7\pi t) + \frac{1}{9}\sin(9\pi t)\right]$$

Result is plotted below: