Solutions to B Problems

CHAPTER 2

B-2-1.

$$f(t) = 0 \qquad t < 0$$
$$= t\,e^{-2t} \qquad t \geq 0$$

Note that

$$\mathcal{L}[t] = \frac{1}{s^2}$$

Referring to Equation (2-2), we obtain

$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}[t\,e^{-2t}] = \frac{1}{(s+2)^2}$$

B-2-2.

(a)

$$f_1(t) = 0 \qquad t < 0$$
$$= 3\sin(5t + 45^\circ) \qquad t \geq 0$$

Note that

$$3\sin(5t + 45^\circ) = 3\sin 5t \cos 45^\circ + 3\cos 5t \sin 45^\circ$$
$$= \frac{3}{\sqrt{2}}\sin 5t + \frac{3}{\sqrt{2}}\cos 5t$$

So we have

$$F_1(s) = \mathcal{L}[f_1(t)] = \frac{3}{\sqrt{2}}\frac{5}{s^2 + 5^2} + \frac{3}{\sqrt{2}}\frac{s}{s^2 + 5^2}$$
$$= \frac{3}{\sqrt{2}}\frac{s+5}{s^2 + 25}$$

(b)

$$f_2(t) = 0 \qquad t < 0$$
$$= 0.03(1 - \cos 2t) \qquad t \geq 0$$

$$F_2(s) = \mathcal{L}[f_2(t)] = 0.03\frac{1}{s} - 0.03\frac{s}{s^2 + 2^2} = \frac{0.12}{s(s^2 + 4)}$$

B-2-3.

$$f(t) = 0 \qquad t < 0$$
$$= t^2 e^{-at} \qquad t \geq 0$$

Note that

$$\mathcal{L}[t^2] = \frac{2}{s^3}$$

Referring to Equation (2-2), we obtain

$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}[t^2 e^{-at}] = \frac{2}{(s + a)^3}$$

B-2-4.

$$f(t) = 0 \qquad t < 0$$
$$= \cos 2\omega t \cos 3\omega t \qquad t \geq 0$$

Noting that

$$\cos 2\omega t \cos 3\omega t = \tfrac{1}{2}(\cos 5\omega t + \cos \omega t)$$

we have

$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}[\tfrac{1}{2}(\cos 5\omega t + \cos \omega t)]$$
$$= \frac{1}{2}\left(\frac{s}{s^2 + 25\omega^2} + \frac{s}{s^2 + \omega^2}\right) = \frac{(s^2 + 13\omega^2)s}{(s^2 + 25\omega^2)(s^2 + \omega^2)}$$

B-2-5. The function f(t) can be written as

$$f(t) = (t - a)\,1(t - a)$$

The Laplace transform of f(t) is

$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}[(t - a)\,1(t - a)] = \frac{e^{-as}}{s^2}$$

B-2-6.

$$f(t) = c\,1(t - a) - c\,1(t - b)$$

The Laplace transform of f(t) is

$$F(s) = c\,\frac{e^{-as}}{s} - c\,\frac{e^{-bs}}{s} = \frac{c}{s}(e^{-as} - e^{-bs})$$

B-2-7. The function f(t) can be written as

$$f(t) = \frac{10}{a^2} - \frac{12.5}{a^2} 1(t - \frac{a}{5}) + \frac{2.5}{a^2} 1(t - a)$$

So the Laplace transform of f(t) becomes

$$F(s) = \mathcal{L}[f(t)] = \frac{10}{a^2}\frac{1}{s} - \frac{12.5}{a^2}\frac{1}{s}e^{-(a/5)s} + \frac{2.5}{a^2}\frac{1}{s}e^{-as}$$

$$= \frac{1}{a^2 s}(10 - 12.5\, e^{-(a/5)s} + 2.5\, e^{-as})$$

As a approaches zero, the limiting value of F(s) becomes as follows:

$$\lim_{a\to 0} F(s) = \lim_{a\to 0} \frac{10 - 12.5\, e^{-(a/5)s} + 2.5\, e^{-as}}{a^2 s}$$

$$= \lim_{a\to 0} \frac{\frac{d}{da}(10 - 12.5\, e^{-(a/5)s} + 2.5\, e^{-as})}{\frac{d}{da} a^2 s}$$

$$= \lim_{a\to 0} \frac{2.5\, s\, e^{-(a/5)s} - 2.5\, s\, e^{-as}}{2as}$$

$$= \lim_{a\to 0} \frac{\frac{d}{da}(2.5\, e^{-(a/5)s} - 2.5\, e^{-as})}{\frac{d}{da} 2a}$$

$$= \lim_{a\to 0} \frac{-0.5\, s\, e^{-(a/5)s} + 2.5\, s\, e^{-as}}{2}$$

$$= \frac{-0.5\, s + 2.5\, s}{2} = \frac{2s}{2} = s$$

B-2-8. The function f(t) can be written as

$$f(t) = \frac{24}{a^3} t - \frac{24}{a^2} 1(t - \frac{a}{2}) - \frac{24}{a^3}(t - a)\, 1(t - a)$$

So the Laplace transform of f(t) becomes

$$F(s) = \frac{24}{a^3}\frac{1}{s^2} - \frac{24}{a^2}\frac{1}{s} e^{-\frac{1}{2}as} - \frac{24}{a^3}\frac{e^{-as}}{s^2}$$

$$= \frac{24}{a^3}\left(\frac{1}{s^2} - \frac{a}{s} e^{-\frac{1}{2}as} - \frac{e^{-as}}{s^2}\right)$$

The limiting value of F(s) as a approaches zero is

$$\lim_{a\to 0} F(s) = \lim_{a\to 0} \frac{24(1 - as\, e^{-\frac{1}{2}as} - e^{-as})}{a^3 s^2}$$

$$= \lim_{a\to 0} \frac{\frac{d}{da} 24(1 - as\, e^{-\frac{1}{2}as} - e^{-as})}{\frac{d}{da} a^3 s^2}$$

$$= \lim_{a\to 0} \frac{24(-s\, e^{-\frac{1}{2}as} + \frac{as^2}{2} e^{-\frac{1}{2}as} + s\, e^{-as})}{3a^2 s^2}$$

$$= \lim_{a\to 0} \frac{\frac{d}{da} 8(-\, e^{-\frac{1}{2}as} + \frac{as}{2} e^{-\frac{1}{2}as} + e^{-as})}{\frac{d}{da} a^2 s}$$

$$= \lim_{a\to 0} \frac{8\left[\frac{s}{2} e^{-\frac{1}{2}as} + \frac{s}{2} e^{-\frac{1}{2}as} + \frac{as}{2}\left(\frac{-s}{2}\right) e^{-\frac{1}{2}as} - se^{-as}\right]}{2as}$$

$$= \lim_{a\to 0} \frac{\frac{d}{da}(4\, e^{-\frac{1}{2}as} - as\, e^{-\frac{1}{2}as} - 4\, e^{-as})}{\frac{d}{da} a}$$

$$= \lim_{a\to 0} \frac{-2s\, e^{-\frac{1}{2}as} - s\, e^{-\frac{1}{2}as} + as\frac{s}{2} e^{-\frac{1}{2}as} + 4s\, e^{-as}}{1}$$

$$= -2s - s + 4s = s$$

<u>B-2-9</u>.

$$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

$$= \lim_{s\to 0} \frac{s\; 5(s + 2)}{s(s + 1)} = \frac{5 \times 2}{1} = 10$$

<u>B-2-10</u>.

$$f(0+) = \lim_{t\to 0+} f(t) = \lim_{s\to\infty} \frac{s\; 2(s + 2)}{s(s + 1)(s + 3)} = 0$$

B-2-11. Define

$$y = \dot{x}$$

Then

$$y(0+) = \dot{x}(0+)$$

The initial value of y can be obtained by use of the initial value theorem as follows:

$$y(0+) = \lim_{s\to\infty} sY(s)$$

Since

$$Y(s) = \mathcal{L}_+[y(t)] = \mathcal{L}_+[\dot{x}(t)] = sX(s) - x(0+)$$

we obtain

$$y(0+) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty} s[sX(s) - x(0+)]$$

$$= \lim_{s\to\infty} [s^2X(s) - sx(0+)]$$

B-2-12. Note that

$$\mathcal{L}\left[\frac{d}{dt} f(t)\right] = sF(s) - f(0)$$

$$\mathcal{L}\left[\frac{d^2}{dt^2} f(t)\right] = s^2F(s) - sf(0) - \dot{f}(0)$$

Define

$$g(t) = \frac{d^2}{dt^2} f(t)$$

Then

$$\mathcal{L}\left[\frac{d^3}{dt^3} f(t)\right] = \mathcal{L}\left[\frac{d}{dt} g(t)\right] = sG(s) - g(0)$$

$$= s[s^2F(s) - sf(0) - \dot{f}(0)] - \ddot{f}(0)$$

$$= s^3F(s) - s^2f(0) - s\dot{f}(0) - \ddot{f}(0)$$

B-2-13.

(a)

$$F_1(s) = \frac{s + 5}{(s + 1)(s + 3)} = \frac{a_1}{s + 1} + \frac{a_2}{s + 3}$$

where

$$a_1 = \left.\frac{s + 5}{s + 3}\right|_{s = -1} = \frac{4}{2} = 2$$

$$a_2 = \left.\frac{s + 5}{s + 1}\right|_{s = -3} = \frac{2}{-2} = -1$$

$F_1(s)$ can thus be written as

$$F_1(s) = \frac{2}{s+1} - \frac{1}{s+3}$$

and the inverse Laplace transform of $F_1(s)$ is

$$f_1(t) = 2\,e^{-t} - e^{-3t}$$

(b)

$$F_2(s) = \frac{3(s+4)}{s(s+1)(s+2)} = \frac{a_1}{s} + \frac{a_2}{s+1} + \frac{a_3}{s+2}$$

where

$$a_1 = \left.\frac{3(s+4)}{(s+1)(s+2)}\right|_{s=0} = \frac{3 \times 4}{2} = 6$$

$$a_2 = \left.\frac{3(s+4)}{s(s+2)}\right|_{s=-1} = \frac{3 \times 3}{(-1) \times 1} = -9$$

$$a_3 = \left.\frac{3(s+4)}{s(s+1)}\right|_{s=-2} = \frac{3 \times 2}{(-2)(-1)} = 3$$

$F_2(s)$ can thus be written as

$$F_2(s) = \frac{6}{s} - \frac{9}{s+1} + \frac{3}{s+2}$$

and the inverse Laplace transform of $F_2(s)$ is

$$f_2(t) = 6 - 9\,e^{-t} + 3\,e^{-2t}$$

B-2-14.

(a)

$$F_1(s) = \frac{6s+3}{s^2} = \frac{6}{s} + \frac{3}{s^2}$$

The inverse Laplace transform of $F_1(s)$ is

$$f_1(t) = 6 + 3t$$

(b)

$$F_2(s) = \frac{5s + 2}{(s + 1)(s + 2)^2} = \frac{a}{s + 1} + \frac{b_2}{(s + 2)^2} + \frac{b_1}{s + 2}$$

where

$$a = \left.\frac{5s + 2}{(s + 2)^2}\right|_{s = -1} = \frac{-5 + 2}{1^2} = -3$$

$$b_2 = \left.\frac{5s + 2}{s + 1}\right|_{s = -2} = \frac{-10 + 2}{-2 + 1} = 8$$

$$b_1 = \left.\frac{d}{ds}\left(\frac{5s + 2}{s + 1}\right)\right|_{s = -2} = \left.\frac{5(s + 1) - (5s + 2)}{(s + 1)^2}\right|_{s = -2}$$

$$= \frac{5(-1) - (-10 + 2)}{1^2} = 3$$

$F_2(s)$ can thus be written as

$$F_2(s) = \frac{-3}{s + 1} + \frac{8}{(s + 2)^2} + \frac{3}{s + 2}$$

and the inverse Laplace transform of $F_2(s)$ is

$$f_2(t) = -3\ e^{-t} + 8t\ e^{-2t} + 3\ e^{-2t}$$

B-2-15.

$$F(s) = \frac{2s^2 + 4s + 5}{s(s + 1)} = 2 + \frac{2}{s + 1} + \frac{5}{s(s + 1)}$$

$$= 2 + \frac{2}{s + 1} + \frac{5}{s} - \frac{5}{s + 1} = 2 - \frac{3}{s + 1} + \frac{5}{s}$$

The inverse Laplace transform of F(s) is

$$f(t) = 2\ \delta(t) - 3\ e^{-t} + 5$$

B-2-16.

$$F(s) = \frac{s^2 + 2s + 4}{s^2} = 1 + \frac{2}{s} + \frac{4}{s^2}$$

The inverse Laplace transform of F(s) is

$$f(t) = \delta(t) + 2 + 4t$$

B-2-17.

$$F(s) = \frac{s}{s^2 + 2s + 10} = \frac{s + 1 - 1}{(s + 1)^2 + 3^2}$$

$$= \frac{s + 1}{(s + 1)^2 + 3^2} - \frac{3}{(s + 1)^2 + 3^2} \frac{1}{3}$$

Hence

$$f(t) = e^{-t} \cos 3t - \frac{1}{3} e^{-t} \sin 3t$$

B-2-18.

$$F(s) = \frac{s^2 + 2s + 5}{s^2(s + 1)} = \frac{a}{s^2} + \frac{b}{s} + \frac{c}{s + 1}$$

where

$$a = \left. \frac{s^2 + 2s + 5}{s + 1} \right|_{s = 0} = 5$$

$$b = \left. \frac{(2s + 2)(s + 1) - (s^2 + 2s + 5)}{(s + 1)^2} \right|_{s = 0} = \frac{2 - 5}{1} = -3$$

$$c = \left. \frac{s^2 + 2s + 5}{s^2} \right|_{s = -1} = \frac{1 - 2 + 5}{1} = 4$$

Hence

$$F(s) = \frac{5}{s^2} + \frac{-3}{s} + \frac{4}{s + 1}$$

The inverse Laplace transform of F(s) is

$$f(t) = 5t - 3 + 4 e^{-t}$$

B-2-19.

$$F(s) = \frac{2s + 10}{(s + 1)^2(s + 4)} = \frac{a}{(s + 1)^2} + \frac{b}{s + 1} + \frac{c}{s + 4}$$

where

$$a = \left. \frac{2s + 10}{s + 4} \right|_{s = -1} = \frac{-2 + 10}{3} = \frac{8}{3}$$

$$b = \left. \frac{2(s + 4) - (2s + 10)}{(s + 4)^2} \right|_{s = -1} = \frac{6 - 8}{3^2} = \frac{-2}{9}$$

$$c = \left. \frac{2s + 10}{(s + 1)^2} \right|_{s = -4} = \frac{-8 + 10}{9} = \frac{2}{9}$$

Hence

$$F(s) = \frac{8}{3} \frac{1}{(s + 1)^2} - \frac{2}{9} \frac{1}{s + 1} + \frac{2}{9} \frac{1}{s + 4}$$

The inverse Laplace transform of F(s) is

$$f(t) = \frac{8}{3} te^{-t} - \frac{2}{9} e^{-t} + \frac{2}{9} e^{-4t}$$

B-2-20.

$$F(s) = \frac{1}{s^2(s^2 + \omega^2)} = \left(\frac{1}{s^2} - \frac{1}{s^2 + \omega^2}\right)\frac{1}{\omega^2}$$

The inverse Laplace transform of F(s) is

$$f(t) = \frac{1}{\omega^2} (t - \frac{1}{\omega} \sin \omega t)$$

B-2-21.

$$F(s) = \frac{c}{s^2} (1 - e^{-as}) - \frac{b}{s} e^{-as} \qquad a > 0$$

The inverse Laplace transform of F(s) is

$$f(t) = ct - c(t - a)1(t - a) - b\, 1(t - a)$$

B-2-22. $\ddot{x} + 4x = 0, \quad x(0) = 5, \quad \dot{x}(0) = 0$

The Laplace transform of the given differential equation is

$$[s^2X(s) - sx(0) - \dot{x}(0)] + 4X(s) = 0$$

Substitution of the initial conditions into this last equation gives

$$(s^2 + 4)X(s) = 5s$$

Solving for X(s), we obtain

$$X(s) = \frac{5s}{s^2 + 4}$$

The inverse Laplace transform of X(s) is

$$x(t) = 5 \cos 2t$$

This is the solution of the given differential equation.

B-2-23. $\ddot{x} + \omega_n^2 x = t, \quad x(0) = 0, \quad \dot{x}(0) = 0$

The Laplace transform of this differential equation is

$$s^2X(s) + \omega_n^2 X(s) = \frac{1}{s^2}$$

Solving this equation for X(s), we obtain

$$X(s) = \frac{1}{s^2(s^2 + \omega_n^2)} = \left(\frac{1}{s^2} - \frac{1}{s^2 + \omega_n^2}\right)\frac{1}{\omega_n^2}$$

The inverse Laplace transform of X(s) is

$$x(t) = \frac{1}{\omega_n^2}\left(t - \frac{1}{\omega_n} \sin \omega_n t\right)$$

This is the solution of the given differential equation.

B-2-24. $2\ddot{x} + 2\dot{x} + x = 1, \quad x(0) = 0, \quad \dot{x}(0) = 2$

The Laplace transform of this differential equation is

$$2[s^2X(s) - sx(0) - \dot{x}(0)] + 2[sX(s) - x(0)] + X(s) = \frac{1}{s}$$