Chapter 1

Introduction

1.1 Problems

1-1. Compute λ_D and N_D for the following cases: (a) A glow discharge, $N_e = 10^{16}$ m^{-3}, $k_B T_e = 2$ eV, (b) The Earth's ionosphere, $N_e = 10^{12}$ m^{-3}, $k_B T_e = 0.1$ eV, (c) A fusion machine, $N_e = 10^{23}$ m^{-3}, $k_B T_e = 9$ keV.

Solution:

$$\lambda_D = \sqrt{\frac{\epsilon_0 k_B T_e}{N_0 q_e^2}}; \qquad N_D = N_0 \left[\frac{4\pi\lambda_D^3}{3}\right]$$

The formulas are straight forward but students often make the mistake of not using SI units for the temperature which requires a conversion from eV to K using 1 eV=11600 K and then multiplying this temperature by $k_B = 1.38 \times 10^{-23}$ Joule/K.

(a)

$$\lambda_D = 1.05 \times 10^{-4} \text{ m}; \qquad N_D = 48738$$

(b)

$$\lambda_D = 0.00235 \text{ m}; \qquad N_D = 54532$$

(c)

$$\lambda_D = 2.23 \times 10^{-6} \text{ m}; \qquad N_D = 4.656 \times 10^6$$

1-2. Calculate the average velocity of nitrogen molecules at room temperature assuming three degrees of freedom.

Solution:

$E_{av} = \frac{3}{2} k_B T$ where $T = 293$ K
$E_{av} = 6.065 \times 10^{-21}$ J; Now use the fact that $E_{av} = \frac{1}{2} m u^2$ where m is the mass of the Nitrogen molecule. Nitrogen gas is diatomic, the molecular mass is 28 g/mol. So

$$m = \frac{28 \text{ g}}{\text{mol}} \cdot \frac{1 \text{ mol}}{6.02 \times 10^{23}} \cdot \frac{1 \text{ kg}}{1000 \text{ g}} = 4.65 \times 10^{-26} \text{ kg}$$

$$u = \sqrt{\frac{2E_{av}}{m}} = 510 \text{ m/s}$$

1-3. Calculate and plot the electrostatic potential and electric field of a test particle of charge $+Q$ in free space and in a plasma of number density N_0 and temperature T. Label the distance axis of your plot in units of Debye length.

Solution:

This problem involves plotting the functions

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

and

$$\Phi(r) = \left[\frac{1}{4\pi\epsilon_0}\frac{Q}{r}\right] e^{-r/\lambda_{\rm D}}$$

Figure 1.1 shows what the plots should look like. The plots shown were obtained using the code shown in Figure 1.2 executed in the MathWorks MATLAB software package.

1-4. A metal sphere of radius, $r = a$, with charge, Q, is placed in a neutral plasma with number density, N_0 and temperature, T. Calculate the effective capacitance of the system. Compare this with the capacitance of the same sphere placed in free space.

Solution:

Capacitance C is the ratio of stored charge Q to potential V: $C = Q/V$. In free space, the potential from a charge Q is given by

$$V(r) = \frac{Q}{4\pi\epsilon_0 r}$$

So the capacitance of the sphere in free space will be

$$C = 4\pi\epsilon_0 a \quad \text{Farads}$$

In a plasma, the potential from the same charge Q will be modified by the Debye shielding effect:

$$V(r) = \frac{1}{4\pi\epsilon_0}\cdot\frac{Q}{r}e^{-r/\lambda_{\rm D}}$$

where

$$\lambda_{\rm D} = \sqrt{\frac{\epsilon_0 k_{\rm B} T_e}{N_0 q_e^2}}$$

Thus the capacitance of the sphere in the plasma will be:

$$C = 4\pi\epsilon_0 a e^{-a/\lambda_{\rm D}} \quad \text{Farads}$$

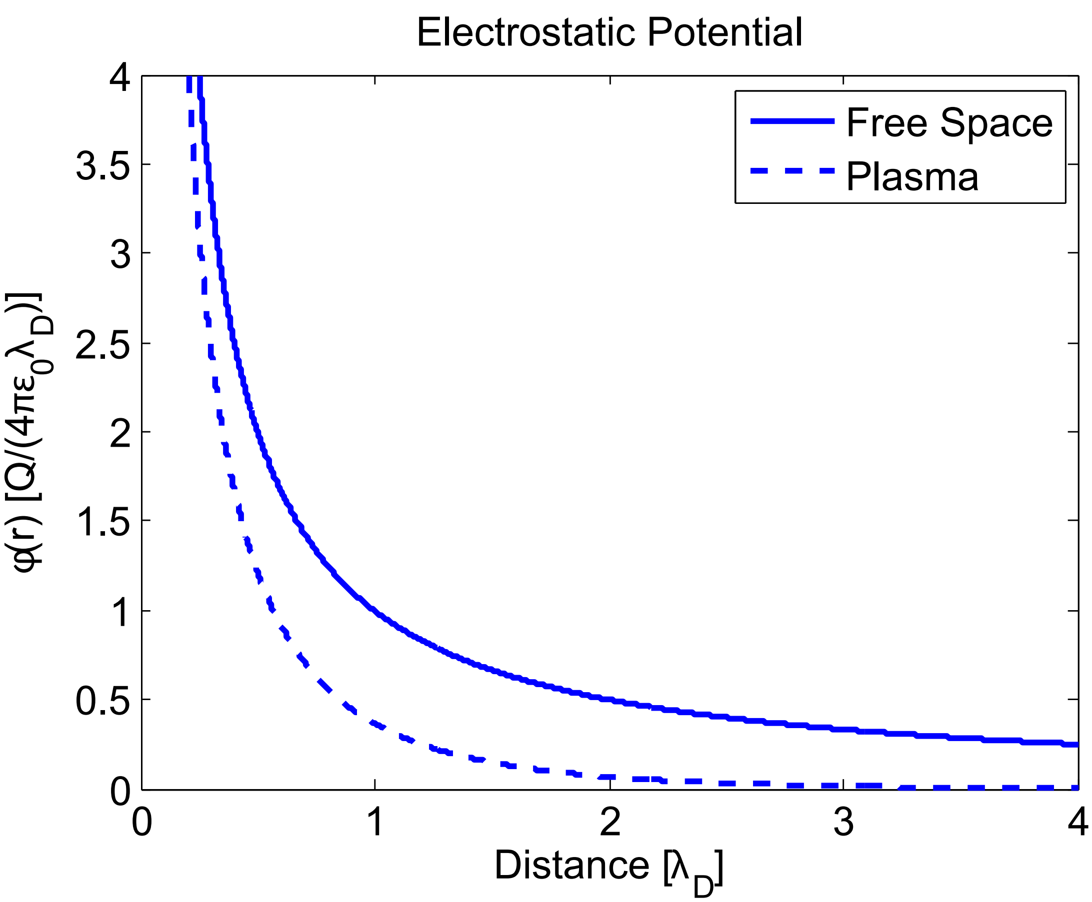

Figure 1.1: Plot for Problem 1-3.

```
%Problem 1-3
%define constants and variables
clear all;
eps0=8.85e-12;
kB=1.38e-23;
qe=1.6e-19;
T=0.2*11600; %Temperature and Density arbitrary for this problem
N0=10^11;
Q=8.65e-12; %Charge also arbitrary

%First caluculate the Debye length
LambdaD=sqrt(eps0*kB*T./(N0*qe.^2));

%Define r vector in terms of Debye length, we will not start at zero
%because the potential goes to infinity at r=0
r=[LambdaD./100:LambdaD./1000:4*LambdaD];

%Define the two potential functions
Phi_freespace=1./(4*pi*eps0).*Q./r;

Phi_plasma=1./(4*pi*eps0).*Q./r.*exp(-r./LambdaD);

%Make the plots, normalize axes to Debye Length and Q/(4*pi*eps0*lambdaD)
figure(1)
plot(r./LambdaD, Phi_freespace.*(4*pi*eps0*LambdaD)/Q,'LineWidth',2); hold on;
plot(r./LambdaD, Phi_plasma.*(4*pi*eps0*LambdaD)/Q, '--', 'LineWidth',2);
hold off

ylim([0 4])
%xlim([0 5e-4])
%make the font size bigger, 'gca' means 'get current axis'
set(gca,'FontSize', 14)

legend('Free Space', 'Plasma')

%Add title and labels
title('Electrostatic Potential')
xlabel('Distance [\lambda_{D}]')
ylabel('\phi(r) [Q/(4\pi\epsilon_0\lambda_D)]')
```

Figure 1.2: MATLAB Code for plot shown in Figure 1.1.

1-5. Consider two infinite, parallel plates parallel plates located at $x = \pm d$, kept at a potential of $\Phi = 0$. The space between the plates is uniformly filled with a gas of density N of particles of charge q. (a) Using Poisson's equation, show that the potential distribution between the plates is $\Phi(x) = [Nq/(2\epsilon_0)](d^2 - x^2)$. (b) Show that for $d > \lambda_{\rm D}$, the energy needed to transport a particle from one of the plates to the mid-point (i.e., $x = 0$) is greater than the average kinetic energy of the particles. (Assume a Maxwellian distribution of particle speeds.)

Solution:

(a) We start with Poisson's equation:

$$\nabla^2\Phi = -\rho/eps_0$$

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = \frac{-Nq}{\epsilon_0}$$

Since the two plates are infinite, this is a one dimensional problem $\frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial z^2} = 0$, Thus:

$$\frac{\partial^2\Phi}{\partial x^2} = \frac{-Nq}{\epsilon_0}$$

$$\frac{\partial\Phi}{\partial x} = \frac{-Nq}{\epsilon_0}x + C_1$$

$$\Phi(x) = -\frac{Nqx^2}{e\epsilon_0} + C_1 x + C_2$$

We now use the boundary condition that the potential at the plates is zero $\Phi(-d) = \Phi(d) = 0$ to get

$$\Phi(x) = \frac{Nq}{2\epsilon_0}\left(d^2 - x^2\right)$$

(b) The energy E_p needed to transport a particle from the wall to the center can be calculated from the difference in potential between these two locations:

$$E_p = q\left(\Phi(0) - \Phi(d)\right)$$

$$E_p = q \cdot \frac{Nq}{2\epsilon_0} \cdot d^2 = \frac{Nq^2d^2}{2\epsilon_0}$$

The average kinetic energy E_k of the particle, assuming a 1-D system is :

$$E_k = \frac{1}{2}k_{\rm B}T$$

We can relate E_p to E_k using the fact that $d > \lambda_{\rm D}$:

$$\begin{aligned}
d > \lambda_{\rm D} &= \sqrt{\frac{\epsilon_0 k_{\rm B} T}{N q^2}} \\
d^2 &> \frac{\epsilon_0 k_{\rm B} T}{N q^2} \\
\frac{N q^2 d^2}{2\epsilon_0} &> \frac{k_{\rm B} T}{2} \\
E_p &> E_k
\end{aligned}$$