2.1

$$a_i = \begin{bmatrix}1\\2\\1\end{bmatrix}, \quad A_{ij} = \begin{bmatrix}1 & 0 & 1\\1 & 2 & 0\\1 & 2 & 1\end{bmatrix}$$

$$a_i a_i = 6 \text{ (scalar)}$$

(a) $$a_i a_j = \begin{bmatrix}1 & 2 & 1\\2 & 4 & 2\\1 & 2 & 1\end{bmatrix} \text{ (matrix)}$$

$$A_{ii} = A_{11} + A_{22} + A_{33} = 4 \text{ (scalar)}$$

$$A_{ij} a_j = \begin{bmatrix}2\\5\\6\end{bmatrix} \text{ (vector)}$$

$$a_i = \begin{bmatrix}0\\2\\1\end{bmatrix}, \quad A_{ij} = \begin{bmatrix}1 & 2 & 1\\1 & 0 & 0\\1 & 2 & 1\end{bmatrix}$$

$$a_i a_i = 5 \text{ (scalar)}$$

(b) $$a_i a_j = \begin{bmatrix}0 & 0 & 0\\0 & 4 & 2\\0 & 2 & 1\end{bmatrix} \text{ (matrix)}$$

$$A_{ii} = A_{11} + A_{22} + A_{33} = 2 \text{ (scalar)}$$

$$A_{ij} a_j = \begin{bmatrix}5\\0\\5\end{bmatrix} \text{ (vector)}$$

$$a_i = \begin{bmatrix}2\\1\\1\end{bmatrix}, \quad A_{ij} = \begin{bmatrix}0 & 2 & 1\\3 & 2 & 0\\1 & 0 & 4\end{bmatrix}$$

$$a_i a_i = 6 \text{ (scalar)}$$

(c) $$a_i a_j = \begin{bmatrix}4 & 2 & 2\\2 & 1 & 1\\2 & 1 & 1\end{bmatrix} \text{ (matrix)}$$

$$A_{ii} = A_{11} + A_{22} + A_{33} = 6 \text{ (scalar)}$$

$$A_{ij} a_j = \begin{bmatrix}3\\8\\6\end{bmatrix} \text{ (vector)}$$

2.2

$$A_{ij} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

(a) $$A_{(ij)} = \frac{1}{2}\left(A_{ij} + A_{ji}\right) = \begin{bmatrix} 1 & 1/2 & 1 \\ 1/2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \text{ six independent components}$$

$$A_{[ij]} = \frac{1}{2}\left(A_{ij} - A_{ji}\right) = \begin{bmatrix} 0 & -1/2 & 0 \\ 1/2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \text{ three independent components}$$

$$A_{ij} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

(b) $$A_{(ij)} = \frac{1}{2}\left(A_{ij} + A_{ji}\right) = \begin{bmatrix} 1 & 3/2 & 1 \\ 3/2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \text{ six independent components}$$

$$A_{[ij]} = \frac{1}{2}\left(A_{ij} - A_{ji}\right) = \begin{bmatrix} 0 & 1/2 & 0 \\ -1/2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \text{ three independent components}$$

$$A_{ij} = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

(c) $$A_{(ij)} = \frac{1}{2}\left(A_{ij} + A_{ji}\right) = \begin{bmatrix} 0 & 5/2 & 1 \\ 5/2 & 2 & 0 \\ 1 & 0 & 4 \end{bmatrix}, \text{ six independent components}$$

$$A_{[ij]} = \frac{1}{2}\left(A_{ij} - A_{ji}\right) = \begin{bmatrix} 0 & -1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ three independent components}$$

2.3.

$$A_{ij}B_{ij} = -A_{ji}B_{ji} = -A_{ij}B_{ij} \Rightarrow 2A_{ij}B_{ij} = 0 \Rightarrow A_{ij}B_{ij} = 0$$

2.4

$$A_{ij}C_{ij} = A_{ji}C_{ij} = A_{ij}C_{ji}$$
$$= A_{ij}\left(C_{(ij)} + C_{[ij]}\right) = A_{ij}C_{(ij)}$$
$$B_{ij}C_{ij} = -B_{ji}C_{ij} = -B_{ij}C_{ji}$$
$$= B_{ij}\left(C_{(ij)} + C_{[ij]}\right) = B_{ij}C_{[ij]}$$

$$\det\left(B_{ij}\right) = \begin{vmatrix} 0 & B_{12} & B_{13} \\ -B_{12} & 0 & B_{23} \\ -B_{13} & -B_{23} & 0 \end{vmatrix} = -B_{12}B_{13}B_{23} + B_{13}B_{12}B_{23} = 0$$

2.5

$$\delta_{ij}a_{jk} = \begin{bmatrix} \delta_{11}a_{11} + \delta_{12}a_{21} + \delta_{13}a_{31} & \delta_{11}a_{12} + \delta_{12}a_{22} + \delta_{13}a_{32} & \delta_{11}a_{13} + \delta_{12}a_{23} + \delta_{13}a_{33} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{ik}$$

$$\delta_{ij}a_{kj} = \begin{bmatrix} \delta_{11}a_{11} + \delta_{12}a_{12} + \delta_{13}a_{13} & \delta_{11}a_{21} + \delta_{12}a_{22} + \delta_{13}a_{23} & \delta_{11}a_{31} + \delta_{12}a_{32} + \delta_{13}a_{33} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = a_{ki}$$

2.6

$$\det(A_{ij}) = \varepsilon_{ijk}A_{1i}A_{2j}A_{3k} = \varepsilon_{123}A_{11}A_{22}A_{33} + \varepsilon_{231}A_{12}A_{23}A_{31} + \varepsilon_{312}A_{13}A_{21}A_{32}$$
$$+ \varepsilon_{321}A_{13}A_{22}A_{31} + \varepsilon_{132}A_{11}A_{23}A_{32} + \varepsilon_{213}A_{12}A_{21}A_{33}$$
$$= A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{13}A_{22}A_{31} - A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33}$$
$$= A_{11}(A_{22}A_{33} - A_{23}A_{32}) - A_{12}(A_{21}A_{33} - A_{23}A_{31}) + A_{13}(A_{21}A_{32} - A_{22}A_{31})$$
$$= \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \text{(using the usual first row expansion)}$$

Likewise the same for the form $\det[A_{ij}] = \varepsilon_{ijk}A_{i1}A_{j2}A_{k3}$

2.7

$$\det[A_{ij}] = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = A_{11}A_{22} - A_{12}A_{21} \ , \ \det[B_{ij}] = \begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{vmatrix} = B_{11}B_{22} - B_{12}B_{21}$$

$$\Rightarrow \det[A_{ij}]\det[B_{ij}] = (A_{11}A_{22} - A_{12}A_{21})(B_{11}B_{22} - B_{12}B_{21})$$

$$= A_{11}A_{22}B_{11}B_{22} + A_{12}A_{21}B_{12}B_{21} - A_{11}A_{22}B_{12}B_{21} - A_{12}A_{21}B_{11}B_{22}$$

$$\det[\boldsymbol{AB}] = \begin{vmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{vmatrix}$$

$$= (A_{11}B_{11} + A_{12}B_{21})(A_{21}B_{12} + A_{22}B_{22}) - (A_{11}B_{12} + A_{12}B_{22})(A_{21}B_{11} + A_{22}B_{21})$$

$$= A_{11}A_{22}B_{11}B_{22} + A_{12}A_{21}B_{12}B_{21} - A_{11}A_{22}B_{12}B_{21} - A_{12}A_{21}B_{11}B_{22} = \det[A_{ij}]\det[B_{ij}]$$

Orthogonality Properties:

Given $A_{ik}A_{jk} = \delta_{ij} \Rightarrow \det(A_{ik}A_{jk}) = \det\delta_{ij} \Rightarrow (\det A)(\det A) = 1 \Rightarrow \det A = \pm 1$

2.8

Property (2.5.4) $\varepsilon_{ijk}\varepsilon_{pqr}\det[A_{mn}] = \begin{vmatrix} A_{ip} & A_{iq} & A_{ir} \\ A_{jp} & A_{jq} & A_{jr} \\ A_{kp} & A_{kq} & A_{kr} \end{vmatrix}$

For the case $A_{ij} = \delta_{ij} \Rightarrow \varepsilon_{ijk}\varepsilon_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix}$ (2.5.5)

From (2.5.5) with $p = i \Rightarrow \varepsilon_{ijk}\varepsilon_{iqr} = \begin{vmatrix} \delta_{ii} & \delta_{iq} & \delta_{ir} \\ \delta_{ji} & \delta_{jq} & \delta_{jr} \\ \delta_{ki} & \delta_{kq} & \delta_{kr} \end{vmatrix} = \delta_{jq}\delta_{kr} - \delta_{jr}\delta_{kq}$ (2.5.6)

From (2.5.6) with $q = j \Rightarrow \varepsilon_{ijk}\varepsilon_{ijr} = \delta_{jj}\delta_{kr} - \delta_{jr}\delta_{kj} = 3\delta_{kr} - \delta_{kr} = 2\delta_{kr}$ (2.5.7)

From (2.5.7) with $r = k \Rightarrow \varepsilon_{ijk}\varepsilon_{ijk} = 2\delta_{kk} = 6$ (2.5.8)

2.9

$\boldsymbol{e}_1 \cdot \boldsymbol{e}_1 = 1 \ , \ \boldsymbol{e}_2 \cdot \boldsymbol{e}_2 = 1 \ , \ \boldsymbol{e}_3 \cdot \boldsymbol{e}_3 = 1$

$\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = \boldsymbol{e}_2 \cdot \boldsymbol{e}_1 = 0 \ , \ \boldsymbol{e}_1 \cdot \boldsymbol{e}_3 = \boldsymbol{e}_3 \cdot \boldsymbol{e}_1 = 0 \ , \ \boldsymbol{e}_2 \cdot \boldsymbol{e}_2 = 1 \ , \ \boldsymbol{e}_3 \cdot \boldsymbol{e}_3 = 1 \Rightarrow \boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$

From (2.6.4) $\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{e}_3 = -\boldsymbol{e}_2 \times \boldsymbol{e}_1 \ , \ \boldsymbol{e}_2 \times \boldsymbol{e}_3 = \boldsymbol{e}_1 = -\boldsymbol{e}_3 \times \boldsymbol{e}_2 \ , \ \boldsymbol{e}_3 \times \boldsymbol{e}_1 = \boldsymbol{e}_2 = -\boldsymbol{e}_3 \times \boldsymbol{e}_1$

$\Rightarrow \ \boldsymbol{e}_i \times \boldsymbol{e}_j = \varepsilon_{ijk}\boldsymbol{e}_k$

2.10

$$\text{For } 90^{o} \text{ rotation about } x_2 - \text{axis}: \; Q_{ij} = \begin{bmatrix} \cos 90^\circ & \cos 90^\circ & \cos 180^\circ \\ \cos 90^\circ & \cos 0^\circ & \cos 90^\circ \\ \cos 0^\circ & \cos 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Cases from Exercise 2.1:

$$\text{(a) } a_i' = Q_{ij}a_j = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \; |\boldsymbol{a}| = (a_i a_i)^{1/2} = \sqrt{6}, \; |\boldsymbol{a}'| = (a_i' a_i')^{1/2} = \sqrt{6}$$

$$A_{ij}' = Q_{ip}Q_{jq}A_{pq} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}, \; A_{ii} = 6, \; A_{ii}' = 6$$

$$\text{(b) } a_i' = Q_{ij}a_j = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \; |\boldsymbol{a}| = (a_i a_i)^{1/2} = \sqrt{6}, \; |\boldsymbol{a}'| = (a_i' a_i')^{1/2} = \sqrt{6}$$

$$A_{ij}' = Q_{ip}Q_{jq}A_{pq} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}, \; A_{ii} = 2, \; A_{ii}' = 2$$

$$\text{(c) } a_i' = Q_{ij}a_j = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \; |\boldsymbol{a}| = (a_i a_i)^{1/2} = \sqrt{6}, \; |\boldsymbol{a}'| = (a_i' a_i')^{1/2} = \sqrt{6}$$

$$A_{ij}' = Q_{ip}Q_{jq}A_{pq} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 2 & 3 \\ -1 & 2 & 0 \end{bmatrix}, \; A_{ii} = 6, \; A_{ii}' = 6$$

2.11

$$\frac{d}{dt}\left(\boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{I}\right) \Rightarrow \dot{\boldsymbol{Q}}\boldsymbol{Q}^T + \boldsymbol{Q}\dot{\boldsymbol{Q}}^T = 0 \Rightarrow$$

$$\dot{\boldsymbol{Q}}\boldsymbol{Q}^T = -\boldsymbol{Q}\dot{\boldsymbol{Q}}^T = -\left(\dot{\boldsymbol{Q}}\boldsymbol{Q}^T\right)^T$$

2.12

$$Q_{ij} = \begin{bmatrix} \cos(x_1', x_1) & \cos(x_1', x_2) \\ \cos(x_2', x_1) & \cos(x_2', x_2) \end{bmatrix} = \begin{bmatrix} \cos\theta & \cos(90^o - \theta) \\ \cos(90^o + \theta) & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$b_i' = Q_{ij} b_j = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_1\cos\theta + b_2 \sin\theta \\ -b_1 \sin\theta + b_2 \cos\theta \end{bmatrix}$$

$$A_{ij}' = Q_{ip} Q_{jq} A_{pq} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^T$$

$$= \begin{bmatrix} A_{11}\cos^2\theta + (A_{12} + A_{21})\sin\theta\cos\theta + A_{22}\sin^2\theta & A_{12}\cos^2\theta - (A_{11} - A_{22})\sin\theta\cos\theta - A_{21}\sin^2\theta \\ A_{21}\cos^2\theta - (A_{11} - A_{22})\sin\theta\cos\theta - A_{12}\sin^2\theta & A_{11}\sin^2\theta - (A_{12} + A_{21})\sin\theta\cos\theta + A_{22}\cos^2\theta \end{bmatrix}$$

2.13

$$a_i = -\frac{1}{2}\varepsilon_{ijk} A_{jk} = -\frac{1}{2}\varepsilon_{ijk}\left(A_{(jk)} + A_{[jk]}\right) = -\frac{1}{2}\varepsilon_{ijk} A_{[jk]} \Rightarrow$$

$$a_i \varepsilon_{imn} = -\frac{1}{2}\varepsilon_{ijk}\varepsilon_{imn} A_{[jk]} = -\frac{1}{2}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) A_{[jk]} = -\frac{1}{2}\left(A_{[mn]} - A_{[nm]}\right) = -A_{[mn]}$$

$$\therefore\ A_{[mn]} = -a_i \varepsilon_{imn}$$

2.14

Property 7 : Given $A_{ij} B_{ij} = c =$ scalar and B_{ij} is a second order tensor $\Rightarrow$

$A_{ij}' B_{ij}' = c' \Rightarrow A_{ij}' Q_{im} Q_{jn} B_{mn} = c = A_{mn} B_{mn} \Rightarrow A_{mn} = Q_{im} Q_{jn} A_{ij}'$ $\therefore$ $\boldsymbol{A}$ is a second order tensor

Property 8 : Given $A_{ij} b_j = c_i$ with b_j and c_i being first order tensors $\Rightarrow$

$A_{ij}' b_j' = c_i' \Rightarrow A_{ij}' Q_{jm} b_m = Q_{in} c_n = Q_{in} A_{nm} b_m \Rightarrow A_{ij}' Q_{jm} = Q_{in} A_{nm} \Rightarrow$

$Q_{km}\left(A_{ij}' Q_{jm} = Q_{in} A_{nm}\right) \Rightarrow A_{ij}' Q_{jm} Q_{km} = Q_{in} A_{nm} Q_{km} \Rightarrow A_{ij}' \delta_{jk} = A_{ik}' = Q_{in} Q_{km} A_{nm}$

$\therefore$ $\boldsymbol{A}$ is a second order tensor

2.15

$$A_{ij}^{-1} A_{jm} = \frac{1}{2\det[\boldsymbol{A}]}\varepsilon_{ikl}\varepsilon_{jqr} A_{qk} A_{rl} A_{jm} = \frac{1}{2\det[\boldsymbol{A}]}\varepsilon_{ikl}\det[\boldsymbol{A}]\varepsilon_{mkl}\text{ , where we have used (2.5.10)}$$

$$= \frac{1}{2}\varepsilon_{ikl}\varepsilon_{mkl} = \frac{1}{2}\varepsilon_{kli}\varepsilon_{klm} = \frac{1}{2}2\delta_{im} = \delta_{im}\text{ , where we have used (2.5.7)}$$

2.16

$I_1 = tr\boldsymbol{A} = I_A$

$I_2 = tr\boldsymbol{A^2} = A_{ij}A_{ji}$

From (2.11.3) $II_A = \frac{1}{2}(A_{ii}A_{jj} - A_{ij}A_{ij}) \Rightarrow A_{ij}A_{ij} = A_{ij}A_{ji} = A_{ii}A_{jj} - 2II_A$

$\therefore I_2 = I_A^2 - 2II_A$

$I_3 = tr\boldsymbol{A^3}$,

Using CT Theorem (2.13.1) $\Rightarrow \boldsymbol{A}^3 = I_A\boldsymbol{A^2} - II_A\boldsymbol{A} + III_A\boldsymbol{I}$

Taking trace operation gives $tr\boldsymbol{A^3} = I_A(I_A^2 - 2II_A) - II_A I_A + 3III_A$

$\therefore I_3 = I_A^3 - 3II_A I_A + 3III_A$

2.17

Given $A_{ij} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$I_A = A_{ii} = A_{11} + A_{22} + A_{33} = \lambda_1 + \lambda_2 + \lambda_3$

$II_A = \frac{1}{2}(A_{ii}A_{jj} - A_{ij}A_{ij}) = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix}$

$= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$

$III_A = \det[A_{ij}] = \lambda_1\lambda_2\lambda_3$

2.18

(a) $A_{ij} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$I_A = A_{ii} = -4$, $II_A = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} = 3$, $III_A = \det[A_{ij}] = 0$

Characteristic Eqn. $-\lambda^3 - 4\lambda^2 - 3\lambda = 0 \Rightarrow \lambda(\lambda+1)(\lambda+3) = 0$, $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = -3$

$\underline{\lambda_1 = 0}$, $(A_{ij} - \lambda\delta_{ij})n_j = 0 \Rightarrow$

$-2n_1^{(1)} + n_2^{(1)} = 0$

$n_1^{(1)} - 2n_2^{(1)} = 0$

$\therefore\ n_1^{(1)} = n_2^{(1)} = 0$ and thus $n_3^{(1)} = 1 \Rightarrow \boldsymbol{n}^{(1)} = \boldsymbol{e}_3$

$\underline{\lambda_2 = -1}$, $(A_{ij} - \lambda\delta_{ij})n_j = 0 \Rightarrow$

$-n_1^{(2)} + n_2^{(2)} = 0$

$n_1^{(2)} - n_2^{(2)} = 0$

$n_3^{(2)} = 0$

$\therefore\ n_1^{(1)} = n_2^{(1)}$ and thus $\Rightarrow \boldsymbol{n}^{(2)} = (\boldsymbol{e}_1 + \boldsymbol{e}_2)/\sqrt{2}$

$\underline{\lambda_3 = -3}$, $(A_{ij} - \lambda\delta_{ij})n_j = 0 \Rightarrow$

$n_1^{(3)} + n_2^{(3)} = 0$

$n_1^{(3)} + n_2^{(3)} = 0$

$3n_3^{(3)} = 0$

$\therefore\ n_1^{(3)} = -n_2^{(3)}$ and thus $\Rightarrow \boldsymbol{n}^{(3)} = (\boldsymbol{e}_1 - \boldsymbol{e}_2)/\sqrt{2}$

Rotation Matrix to Principal Axes: $Q_{ij} = \begin{bmatrix} 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \Rightarrow$

$$\boldsymbol{A}' = \boldsymbol{QAQ^T} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

2.18

(b) $A_{ij} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$$I_A = A_{ii} = 0\ ,\ II_A = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} = -4\ ,\ III_A = \det[A_{ij}] = 0$$

Characteristic Eqn. $-\lambda^3 + 4\lambda = 0 \Rightarrow \lambda(\lambda^2 - 4) = 0,\ \lambda_1 = 0,\ \lambda_2 = 2,\ \lambda_3 = -2$

$\underline{\lambda_1 = 0}$, $(A_{ij} - \lambda\delta_{ij})n_j = 0 \Rightarrow$

$-n_1^{(1)} + n_2^{(1)} = 0$

$n_1^{(1)} - n_2^{(1)} = 0$

$2n_3^{(1)} = 0 \Rightarrow n_3^{(1)} = 0$

$\therefore\ n_1^{(1)} = n_2^{(1)}$ and normalizing $n_1^{(1)} = n_2^{(1)} = 1/\sqrt{2} \Rightarrow \boldsymbol{n}^{(1)} = (\boldsymbol{e}_1 + \boldsymbol{e}_2)/\sqrt{2}$

$\underline{\lambda_2 = -2}$, $(A_{ij} - \lambda\delta_{ij})n_j = 0 \Rightarrow$

$n_1^{(2)} + n_2^{(2)} = 0$

$n_1^{(2)} + n_2^{(2)} = 0$

$4n_3^{(2)} = 0 \Rightarrow n_3^{(2)} = 0$

$\therefore\ n_1^{(2)} = -n_2^{(2)}$ and thus $\Rightarrow \boldsymbol{n}^{(3)} = (\boldsymbol{e}_1 - \boldsymbol{e}_2)/\sqrt{2}$

$\underline{\lambda_3 = 2}$, $(A_{ij} - \lambda\delta_{ij})n_j = 0 \Rightarrow$

$-3n_1^{(3)} + n_2^{(3)} = 0$

$n_1^{(3)} - 3n_2^{(3)} = 0$

$0 = 0 \Rightarrow$

$\therefore\ n_1^{(3)} = n_2^{(3)} = 0$ and thus $n_3^{(3)} = 1 \Rightarrow \boldsymbol{n}^{(3)} = \boldsymbol{e}_3$

Rotation Matrix to Principal Axes: $Q_{ij} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$

$$\boldsymbol{A}' = \boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

2.18

(c) $A_{ij} = \begin{bmatrix} 6 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

$I_A = A_{ii} = 18\ ,\ II_A = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} = 99\ ,\ III_A = \det[A_{ij}] = 162$

Characteristic Eqn. $-\lambda^3 + 18\lambda^2 - 99\lambda + 162 = 0 \Rightarrow (\lambda-9)(\lambda-6)(\lambda-3) = 0 \Rightarrow \lambda_1 = 9,\ \lambda_2 = 6,\ \lambda_3 = 3$

$\underline{\lambda_1 = 9}$, $(A_{ij} - \lambda\delta_{ij})n_j = 0 \Rightarrow$

$-3n_1^{(1)} - 3n_2^{(1)} = 0$

$-3n_1^{(1)} - 3n_2^{(1)} = 0$

$-3n_3^{(1)} = 0 \Rightarrow n_3^{(1)} = 0$

$\therefore\ n_1^{(1)} = -n_2^{(1)}$ and normalizing $\boldsymbol{n}^{(1)} = (\boldsymbol{e}_1 - \boldsymbol{e}_2)/\sqrt{2}$

$\underline{\lambda_2 = 6}$, $(A_{ij} - \lambda\delta_{ij})n_j = 0 \Rightarrow$

$-3n_2^{(2)} = 0 \Rightarrow n_2^{(2)} = 0$

$-3n_1^{(2)} = 0 \Rightarrow n_1^{(2)} = 0$

$0 = 0$

$\therefore\ n_3^{(2)} = 1$ and thus $\Rightarrow \boldsymbol{n}^{(3)} = \boldsymbol{e}_3$

$\underline{\lambda_3 = 3}$, $(A_{ij} - \lambda\delta_{ij})n_j = 0 \Rightarrow$

$3n_1^{(3)} - 3n_2^{(3)} = 0$

$-3n_1^{(3)} + 3n_2^{(3)} = 0$

$3n_3^{(3)} = 0 \Rightarrow n_3^{(3)} = 0$

$\therefore\ n_1^{(3)} = n_2^{(3)}$ and normalizing $n_1^{(1)} = n_2^{(1)} = 1/\sqrt{2} \Rightarrow \boldsymbol{n}^{(3)} = (\boldsymbol{e}_1 + \boldsymbol{e}_2)/\sqrt{2}$

Rotation Matrix to Principal Axes: $Q_{ij} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \Rightarrow$

$\boldsymbol{A}' = \boldsymbol{QAQ^T} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

2.19

Mulitply basic equation $\boldsymbol{An} = \lambda \boldsymbol{n}$ by $\boldsymbol{A}^{-1}$ to get

$$\boldsymbol{A}^{-1}\boldsymbol{An} = \boldsymbol{A}^{-1}\lambda\boldsymbol{n} \Rightarrow \boldsymbol{n} = \lambda\boldsymbol{A}^{-1}\boldsymbol{n} \Rightarrow \boldsymbol{A}^{-1}\boldsymbol{n} = \frac{1}{\lambda}\boldsymbol{n}$$

$\therefore 1/\lambda$ are the principal values of $\boldsymbol{A}^{-1}$, and $\boldsymbol{n}$ are the principal directions

Using principal coordinates,

$$I_A = \lambda_1 + \lambda_2 + \lambda_3 \text{ , } II_A = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \text{ , } III_A = \lambda_1\lambda_2\lambda_3 \Rightarrow$$

$$II_A / III_A = \frac{\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1}{\lambda_1\lambda_2\lambda_3} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = I_{A^{-1}}$$

2.20

Using principal axes $A_{ij}b_ib_j = \lambda_1 b_1^2 + \lambda_2 b_2^2 + \lambda_3 b_3^2$

If $\lambda_i \geq 0$, then clearly $A_{ij}b_ib_j \geq 0$

If $A_{ij}b_ib_j \geq 0 \Rightarrow \lambda_1 b_1^2 + \lambda_2 b_2^2 + \lambda_3 b_3^2 \geq 0$

Since vector $\boldsymbol{b}$ is arbitrary, choose $\boldsymbol{b} = (1,0,0) \Rightarrow \lambda_1 \geq 0$

Likewise for other choices $\boldsymbol{b} = (0,1,0) \Rightarrow \lambda_2 \geq 0$, $\boldsymbol{b} = (0,0,1) \Rightarrow \lambda_3 \geq 0$

2.21

(a) $$A_{ij} = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 2 & 4 \end{bmatrix} \Rightarrow A_{mm} = 9$$

$$\tilde{A}_{ij} = \frac{1}{3} A_{mm} \delta_{ij} = 3\delta_{ij} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\hat{A}_{ij} = A_{ij} - \tilde{A}_{ij} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

(b) $$A_{ij} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix} \Rightarrow A_{mm} = 3$$

$$\tilde{A}_{ij} = \frac{1}{3} A_{mm} \delta_{ij} = \delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\hat{A}_{ij} = A_{ij} - \tilde{A}_{ij} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

(c) $$A_{ij} = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 4 \end{bmatrix} \Rightarrow A_{mm} = 6$$

$$\tilde{A}_{ij} = \frac{1}{3} A_{mm} \delta_{ij} = 2\delta_{ij} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\hat{A}_{ij} = A_{ij} - \tilde{A}_{ij} = \begin{bmatrix} -2 & 2 & 1 \\ 3 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

2.22

From (2.13.2) $\Rightarrow \boldsymbol{A}^3 = I_A\boldsymbol{A}^2 - II_A\boldsymbol{A} + III_A\boldsymbol{I}$

Multiply by $\boldsymbol{A} \Rightarrow \boldsymbol{A}^4 = I_A\boldsymbol{A}^3 - II_A\boldsymbol{A}^2 + III_A\boldsymbol{A}$; substituting in for $\boldsymbol{A}^3$ from (2.13.2) $\Rightarrow$

$$\boldsymbol{A}^4 = I_A(I_A\boldsymbol{A}^2 - II_A\boldsymbol{A} + III_A\boldsymbol{I}) - II_A\boldsymbol{A}^2 + III_A\boldsymbol{A} = (I_A^2 - II_A)\boldsymbol{A}^2 + (III_A - I_AII_A)\boldsymbol{A} + I_AIII_A\boldsymbol{I}$$

Next multiplying by $\boldsymbol{A}$ again $\Rightarrow$

$\boldsymbol{A}^5 = (I_A^2 - II_A)\boldsymbol{A}^3 + (III_A - I_AII_A)\boldsymbol{A}^2 + I_AIII_A\boldsymbol{A}$; substituting in for $\boldsymbol{A}^3$ from (2.13.2) $\Rightarrow$

$$\boldsymbol{A}^5 = (I_A^2 - II_A)(I_A\boldsymbol{A}^2 - II_A\boldsymbol{A} + III_A\boldsymbol{I}) + (III_A - I_AII_A)\boldsymbol{A}^2 + I_AIII_A\boldsymbol{A}$$

$$\boldsymbol{A}^5 = (I_A^3 - 2I_AII_A + III_A)\boldsymbol{A}^2 + (I_AIII_A - I_A^2II_A + II_A^2)\boldsymbol{A} + (I_A^2III_A - II_AIII_A)\boldsymbol{I}$$

2.23

$\boldsymbol{A} = \boldsymbol{RU} = \boldsymbol{VR} \Rightarrow$

$$\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{U}^T\boldsymbol{R}^T\boldsymbol{RU} = \boldsymbol{U}^T\boldsymbol{IU} = \boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{U}^2$$

$$\boldsymbol{AA}^T = \boldsymbol{VRR}^T\boldsymbol{V}^T = \boldsymbol{VI}\,\boldsymbol{V}^T = \boldsymbol{V}\,\boldsymbol{V}^T = \boldsymbol{V}\,\boldsymbol{V} = \boldsymbol{V}^2$$

2.24

(a) $\boldsymbol{u} = x_1\boldsymbol{e_1} + x_1x_2\boldsymbol{e_2} + 2x_1x_2x_3\boldsymbol{e_3}$

$\nabla\cdot\boldsymbol{u} = u_{1,1} + u_{2,2} + u_{3,3} = 1 + x_1 + 2x_1x_2$

$$\nabla\times\boldsymbol{u} = \begin{vmatrix} \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1 & x_1x_2 & 2x_1x_2x_3 \end{vmatrix} = 2x_1x_3\boldsymbol{e_1} - 2x_2x_3\boldsymbol{e_2} + x_2\boldsymbol{e_3}$$

$\nabla^2\boldsymbol{u} = \nabla^2u_1\boldsymbol{e_1} + \nabla^2u_2\boldsymbol{e_2} + \nabla^2u_3\boldsymbol{e_3} = 0\boldsymbol{e_1} + 0\boldsymbol{e_2} + 0\boldsymbol{e_3} = 0$

$$\nabla\boldsymbol{u} = \begin{bmatrix} 1 & 0 & 0 \\ x_2 & x_1 & 0 \\ 2x_2x_3 & 2x_1x_3 & 2x_1x_2 \end{bmatrix},\ tr(\nabla\boldsymbol{u}) = 1 + x_1 + 2x_1x_2$$

(b) $\boldsymbol{u} = x_1^2\boldsymbol{e_1} + 2x_1x_2\boldsymbol{e_2} + x_3^3\boldsymbol{e_3}$

$\nabla\cdot\boldsymbol{u} = u_{1,1} + u_{2,2} + u_{3,3} = 2x_1 + 2x_1 + 3x_3^2$

$$\nabla\times\boldsymbol{u} = \begin{vmatrix} \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1^2 & 2x_1x_2 & x_3^3 \end{vmatrix} = 0\boldsymbol{e_1} - 0\boldsymbol{e_2} + 2x_2\boldsymbol{e_3}$$

$\nabla^2\boldsymbol{u} = \nabla^2u_1\boldsymbol{e_1} + \nabla^2u_2\boldsymbol{e_2} + \nabla^2u_3\boldsymbol{e_3} = 2\boldsymbol{e_1} + 0\boldsymbol{e_2} + 6x_3\boldsymbol{e_3} = 0$

$$\nabla\boldsymbol{u} = \begin{bmatrix} 2x_1 & 0 & 0 \\ 2x_2 & 2x_1 & 0 \\ 0 & 0 & 3x_3^2 \end{bmatrix},\ tr(\nabla\boldsymbol{u}) = 4x_1 + 3x_3^2$$

(c) $\boldsymbol{u} = x_2^2\boldsymbol{e_1} + 2x_2x_3\boldsymbol{e_2} + 4x_1^2\boldsymbol{e_3}$

$\nabla\cdot\boldsymbol{u} = u_{1,1} + u_{2,2} + u_{3,3} = 0 + 2x_3 + 0$

$$\nabla\times\boldsymbol{u} = \begin{vmatrix} \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_2^2 & 2x_2x_3 & 4x_1^2 \end{vmatrix} = -2x_2\boldsymbol{e_1} - 8x_1\boldsymbol{e_2} - 2x_2\boldsymbol{e_3}$$

$\nabla^2\boldsymbol{u} = \nabla^2u_1\boldsymbol{e_1} + \nabla^2u_2\boldsymbol{e_2} + \nabla^2u_3\boldsymbol{e_3} = 2\boldsymbol{e_1} + 0\boldsymbol{e_2} + 8\boldsymbol{e_3}$

$$\nabla\boldsymbol{u} = \begin{bmatrix} 0 & 2x_2 & 0 \\ 0 & 2x_3 & 2x_2 \\ 8x_1 & 0 & 0 \end{bmatrix},\ tr(\nabla\boldsymbol{u}) = 2x_3$$

2.25

(a)

$$\nabla(\phi\psi)=(\phi\psi)_{,k}=\phi\psi_{,k}+\phi_{,k}\psi=\nabla\phi\psi+\phi\nabla\psi$$

$$\nabla^2(\phi\psi)=(\phi\psi)_{,kk}=(\phi\psi_{,k}+\phi_{,k}\psi)_{,k}=\phi\psi_{,kk}+\phi_{,k}\psi_{,k}+\phi_{,k}\psi_{,k}+\phi_{,kk}\psi=\phi_{,kk}\psi+\phi\psi_{,kk}+2\phi_{,k}\psi_{,k}$$

$$=(\nabla^2\phi)\psi+\phi(\nabla^2\psi)+2\nabla\phi\cdot\nabla\psi$$

$$\nabla\cdot(\phi\boldsymbol{u})=(\phi u_k)_{,k}=\phi u_{k,k}+\phi_{,k}u_k=\nabla\phi\cdot\boldsymbol{u}+\phi(\nabla\cdot\boldsymbol{u})$$

(b)

$$\nabla\times(\phi\boldsymbol{u})=\varepsilon_{ijk}(\phi u_k)_{,j}=\varepsilon_{ijk}(\phi u_{k,j}+\phi_{,j}u_k)=\varepsilon_{ijk}\phi_{,j}u_k+\phi\varepsilon_{ijk}u_k=\nabla\phi\times\boldsymbol{u}+\phi(\nabla\times\boldsymbol{u})$$

$$\nabla\cdot(\boldsymbol{u}\times\boldsymbol{v})=(\varepsilon_{ijk}u_jv_k)_{,i}=\varepsilon_{ijk}(u_jv_{k,i}+u_{j,i}v_k)=v_k\varepsilon_{ijk}u_{j,i}+u_j\varepsilon_{ijk}v_{k,i}=\boldsymbol{v}\cdot(\nabla\times\boldsymbol{u})-\boldsymbol{u}\cdot(\nabla\times\boldsymbol{v})$$

$$\nabla\times\nabla\phi=\varepsilon_{ijk}(\phi_{,k})_{,j}=\varepsilon_{ijk}\phi_{,kj}=0\ \text{because of symmetry and antisymmetry in } jk$$

$$\nabla\cdot\nabla\phi=(\phi_{,k})_{,k}=\phi_{,kk}=\nabla^2\phi$$

(c)

$$\nabla\cdot(\nabla\times\boldsymbol{u})=(\varepsilon_{ijk}u_{k,j})_{,i}=\varepsilon_{ijk}u_{k,ji}=0,\ \text{because of symmetry and antisymmetry in } ij$$

$$\nabla\times(\nabla\times\boldsymbol{u})=\varepsilon_{mni}(\varepsilon_{ijk}u_{k,j})_{,n}=\varepsilon_{imn}\varepsilon_{ijk}u_{k,jn}=(\delta_{mj}\delta_{nk}-\delta_{mk}\delta_{nj})u_{k,jn}=u_{n,nm}-u_{m,nn}$$

$$=\nabla(\nabla\cdot\boldsymbol{u})-\nabla^2\boldsymbol{u}$$

$$\boldsymbol{u}\times(\nabla\times\boldsymbol{u})=\varepsilon_{ijk}u_j(\varepsilon_{kmn}u_{n,m})=\varepsilon_{kij}\varepsilon_{kmn}u_ju_{n,m}=(\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm})u_ju_{n,m}=u_nu_{n,i}-u_mu_{i,m}$$

$$=\frac{1}{2}\nabla(\boldsymbol{u}\cdot\boldsymbol{u})-\boldsymbol{u}\cdot\nabla\boldsymbol{u}$$

2.26

$$\text{Cylindrical coordinates}:\ \xi^1=r,\ \xi^2=\theta,\ \xi^3=z$$

$$(ds)^2=(dr)^2+(rd\theta)^2+(dz)^2\ \Rightarrow\ h_1=1,\ h_2=r,\ h_3=1$$

$$\hat{\boldsymbol{e}}_r=\cos\theta\boldsymbol{e}_1+\sin\theta\boldsymbol{e}_2,\ \hat{\boldsymbol{e}}_\theta=-\sin\theta\boldsymbol{e}_1+\cos\theta\boldsymbol{e}_2,\ \hat{\boldsymbol{e}}_z=\boldsymbol{e}_3$$

$$\frac{\partial\hat{\boldsymbol{e}}_r}{\partial\theta}=\hat{\boldsymbol{e}}_\theta,\ \frac{\partial\hat{\boldsymbol{e}}_\theta}{\partial\theta}=-\hat{\boldsymbol{e}}_r,\ \frac{\partial\hat{\boldsymbol{e}}_r}{\partial r}=\frac{\partial\hat{\boldsymbol{e}}_\theta}{\partial r}=\frac{\partial\hat{\boldsymbol{e}}_z}{\partial r}=\frac{\partial\hat{\boldsymbol{e}}_z}{\partial\theta}=\frac{\partial\hat{\boldsymbol{e}}_z}{\partial z}=0$$

$$\nabla=\hat{\boldsymbol{e}}_r\frac{\partial}{\partial r}+\hat{\boldsymbol{e}}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+\hat{\boldsymbol{e}}_z\frac{\partial}{\partial z}$$

$$\nabla f=\hat{\boldsymbol{e}}_r\frac{\partial f}{\partial r}+\hat{\boldsymbol{e}}_\theta\frac{1}{r}\frac{\partial f}{\partial\theta}+\hat{\boldsymbol{e}}_z\frac{\partial f}{\partial z}$$

$$\nabla\cdot\boldsymbol{u}=\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial u_z}{\partial z}$$

$$\nabla^2f=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2f}{\partial\theta^2}+\frac{\partial^2f}{\partial z^2}$$

$$\nabla\times\boldsymbol{u}=\left(\frac{1}{r}\frac{\partial u_z}{\partial\theta}-\frac{\partial u_\theta}{\partial z}\right)\hat{\boldsymbol{e}}_r+\left(\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right)\hat{\boldsymbol{e}}_\theta+\frac{1}{r}\left(\frac{\partial}{\partial r}(ru_\theta)-\frac{\partial u_r}{\partial\theta}\right)\hat{\boldsymbol{e}}_z$$