C H A P T E R

2
Optimum Design Problem Formulation

2.1————————————————————————————

A 100 ×100 m lot is available to construct a multistory office building. At least 20,000 m^2 total floor space is needed. According to a zoning ordinance, the maximum height of the building can be only 21 m, and the area for parking outside the building must be at least 25 percent of the total floor area of all the stories. It has been decided to fix the height of each story at 3.5 m. The cost of the building in millions of dollars is estimated at 0.6*h* +0.001*A*, where *A* is the cross-sectional area of the building per floor and *h* is the height of the building. Formulate the minimum cost design problem.

Solution

Given: The lot size, building floor space and parking area requirements, and the data given in the problem statement.

Required: It is desired to find the building cross-sectional area and its height to meet all the requirements and minimize cost of the building.

Procedure: We follow the five step process to formulate the problem as an optimization problem.

Step 1: Project/Problem Statement
Shown above

Step 2: Data and Information Collection
Area of the lot =100×100 = 10,000 m^2
Area available for parking = (10,000 – *A*), m^2
Total floor area = (number of floors)×$A = \dfrac{h}{3.5}A$, m^2

Step 3: Definition of Design Variables
A = cross-sectional area of the building for each floor, m^2
h = height of the building, m

Step 4: Optimization Criterion
Optimization criterion is to minimize $ cost, and the cost function is defined as
Cost = (0.6*h* + 0.001*A*), million dollars   (1)

Step 5: Formulation of Constraints
Floor Space Constraint:
  $hA/3.5 \geq 20{,}000,\ \text{m}^2$ (2)
Parking Constraint:
  $(10{,}000 - A) \geq 0.25hA/3.5,\ \text{m}^2$ (3)

Explicit Design Variable Constraints:

$$h \geq 3.5, \text{ m} \qquad\qquad (4)$$
$$h \leq 21, \text{ m} \qquad\qquad (5)$$
$$A \geq 0, \text{ m}^2 \qquad\qquad (6)$$
$$A \leq 10000, \text{ m}^2 \qquad\qquad (7)$$

Final Formulation:

Find h and A to minimize the cost function of Eq. (1) subject to the constraints in Eqs. (2) to (7). Note that for a meaningful design, h must be a multiple of 3.5.

2.2————————————————————————————————————

A refinery has two crude oils:

1. Crude *A* costs $120/barrel (bbl) and 20,000 bbl are available.
2. Crude *B* costs $150/bbl and 30,000 bbl are available.

The company manufactures gasoline and lube oil from the crudes. Yield and sale price barrel of the product and markets are shown in Table E2.2. How much crude oils should the company use to maximize its profit? Formulate the optimum design problem.

Table E2.2 Data for Refinery Operation

Product	Yield/bbl		Sale Price per bbl ($)	Market (bbl)
	Crude *A*	Crude *B*		
Gasoline	0.6	0.8	200	20,000
Lube oil	0.4	0.2	450	10,000

Solution

Given: The cost of two crude oils per barrel, the amount of barrels available for each type, and all information shown in Table E2.2.

Required: It is desired to find the amount of each crude oil which should be used, subject to the above constraints, to maximize profit.

Procedure: We follow the five step process to formulate the problem as an optimization problem.

Step 1: Problem Statement
    Shown above

Step 2: Data and Information Collection
    Shown above

Step 3: Definition of Design Variables
    A = *Crude A* used in barrels
    B = *Crude B* used in barrels

Step 4: Optimization Criterion
    Optimization criterion is to maximize profit, and the cost function is defined as
    $Profit = 200(0.6A + 0.8B) + 450(0.4A + 0.2B) - 120A - 150B = 180A + 100B$

Step 5: Formulation of Constraints
    Gasoline Market Constraint:
            $(0.6A + 0.8B) \leq 20{,}000$, bbl

    Lube Oil Market Constraint:
            $(0.4A + 0.2B) \leq 10{,}000$, bbl

Explicit Design Variable Constraints:

$A \leq 20,000$, bbl

$B \leq 30,000$, bbl

$A \geq 0; \ B \geq 0$

2.3

Design a beer bug, shown in Fig. E2.3, to hold as much beer as possible. The height and radius of the mug should be not more than 20 cm. The mug must be at least 5 cm in radius. The surface area of the sides must not be greater than 900 cm^2 (ignore the area of the bottom of the mug and ignore the mug handle – see figure). Formulate the optimum design problem.
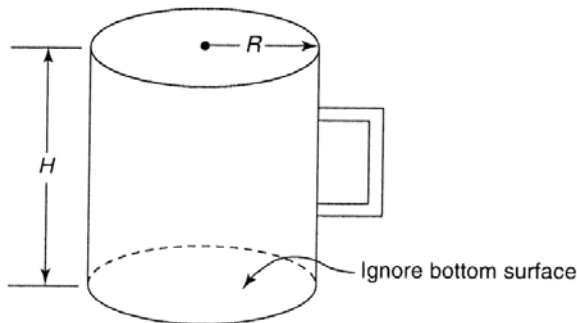

FIGURE E2.3 Beer mug.

Solution

Given: The maximum and minimum radius of the mug, the maximum height of the mug, and the maximum surface area of the mug. The area of the bottom of the mug is ignored.

Required: It is desired to find the dimensions of the beer mug which will maximize the amount of beer it can hold.

Procedure: We follow the five step process to formulate the problem as an optimization problem.

Step 1: Problem Statement
Shown above

Step 2: Data and Information Collection
Shown above

Step 3: Definition of Design Variables
R = radius of the mug in cm
H = height of the mug in cm

Step 4: Optimization Criterion
Optimization criterion is to maximize volume of the mug, and the cost function is defined as
$Volume = \pi R^2 H,$ cm^3

Step 5: Formulation of Constraints
Surface Area Constraint: $2\pi RH \leq 900,$ cm²
Explicit Design Variable Constraints:
$R \geq 5$ cm, $R \leq 20$ cm; $H \geq 0$ cm, $H \leq 20$ cm

2.4———————————————————————————

A company is redesigning its parallel flow heat exchanger of length *l* to increase its heat transfer. An end view of the units is shown in Fig. E2.4. There are certain limitations on the design problem. The smallest available conducting tube has a radius of 0.5 cm and all tubes must be of the same size. Further, the total cross sectional area of all the tubes cannot exceed 2000 cm^2 to ensure adequate space inside the outer shell. Formulate the problem to determine the number of tubes and the radius of each tube to maximize the surface area of the tubes in the exchanger.
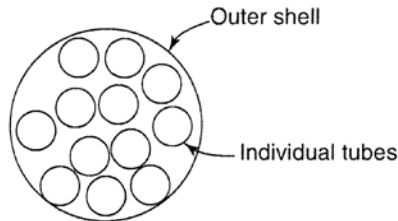

FIGURE E2.4 Cross section of heat exchanger.

Solution

Given: The minimum radius of each tube, the similarity between each tube, and the maximum surface area of all tubes combined.

Required: It is desired to find the number of tubes and the radius of each tube which will maximize the surface area of the tubes in the heat exchanger.

Procedure: We follow the five step process to formulate the problem as an optimization problem.

Step 1: Problem Statement
Shown above

Step 2: Data and Information Collection
Shown above

Step 3: Definition of Design Variables
N = number of tubes
R = radius of the tubes, cm

Step 4: Optimization Criterion
Optimization criterion is to maximize surface area of tubes, and the cost function is defined as
Surface area $= N(2\pi R)l = 2\pi RlN$, cm^2

Step 5: Formulation of Constraints
Cross-sectional Area Constraint: $N(\pi R^2) \leq 2000$, cm^2
Explicit Design Variable Constraints:
$R \geq 0.5$, cm; $N \geq 0$

Note that for a meaningful solution, N should assume an integer value.

2.5

Proposals for a parking ramp have been defeated, so we plan to build parking lot in the downtown urban renewal section. The cost of land is $200W + 100D$, where W is the width along the street and D the depth of the lot in meters. The available width along the street is 100 m, while the maximum depth available is 200 m. We want to have at least 10,000 m^2 in the lot. To avoid unsightliness, the city requires that the longer dimension of any lot be no more than twice the shorter dimension. Formulate the minimum-cost design problem.

<u>Solution</u>

Given: The cost of land in the downtown urban renewal section, the maximum width and depth available, and the minimum area available in the lot. In addition, the longer dimension can be no more than twice the shorter dimension.

Required: Minimize the cost required to build such a parking lot, subject to the given constraints.

Procedure: We follow the five step process to formulate the problem as an optimization problem.

Step 1: Problem Statement
Shown above

Step 2: Data and Information Collection
Shown above

Step 3: Definition of Design Variables
W = width of lot in m
D = depth of lot in m

Step 4: Optimization Criterion
Optimization criterion is to minimize $ cost, and the cost function is defined as
$Cost = 200W + 100D$, $

Step 5: Formulation of Constraints
Width Limitation Constraint: $W \leq 100$, m
Depth Limitation Constraint: $D \leq 200$, m
Area Constraint: $WD \geq 10000$
Explicit Design Variable Constraints:
$D \leq 2W$, m
$W \leq 2D$, m
$W \geq 0$, m
$D \geq 0$, m

2.6———————————————————————————————————

A manufacturer sells products A and B. Profit from A is $10/kg and is $8/kg from B. Available raw materials for the products are 100 kg of C and 80 kg of D. To produce 1 kg of A, we need 0.4 kg of C and 0.6kg of D. To produce 1 kg of B, we need 0.5 kg of C and 0.5 kg of D. The markets for the products are 70 kg for A and 110 kg for B. How much A and B should be produced to maximize profit? Formulate the design optimization problem.

Solution

Given: The profits from selling products A and B, the amount of raw material available of products C and D, the amount of products C and D required to produce products A and B, and the market for products A and B.
Required: It is desired to find the amount of A and B which should be produced to maximize profit.
Procedure: We follow the five step process to formulate the problem as an optimization problem.

Step 1: Problem Statement
    Shown above

Step 2: Data and Information Collection
    Shown above

Step 3: Definition of Design Variables
    A = product A produced in kg
    B = product B produced in kg

Step 4: Optimization Criterion
    Optimization criterion is to maximize profit, and the cost function is defined as
    $Profit = 10A + 8B$, $ (1)

Step 5: Formulation of Constraints
    Limits on Products Constraints:
        $A \leq 70$ kg, $B \leq 110$ kg   (2)
    Raw Material Constraints:
        Amount of C used to produce A and B: $C = 0.4A + 0.5B$ (3)
        Amount of D used to produce A and B: $D = 0.6A + 0.5B$ (4)

        Constraint on C used: $0.4A + 0.5B \leq 100$ kg   (5)
        Constraint on D used: $0.6A + 0.5B \leq 80$ kg   (6)
    Explicit Design Variable Constraints:
        $A \geq 0, B \geq 0$ (7)
    Final formulation: Find A and B to maximize the profit function in Eq. (1), subject to constraints in Eqs (5) – (7).

Alternate Formulation

If C (amount of C used in kg) and D (amount of D used in kg) are also treated as design variables, then the formulation of the problem will become:

Formulation: Find $A,\ B,\ C$, and D to maximize the profit in Eq. (1) subject to the constraints:

Limits on variables:
$$A \le 70 \text{ kg}, \quad B \le 110 \text{ kg}, \quad C \le 100 \text{ kg}, \quad D \le 80 \text{ kg} \tag{8}$$

Raw Material Constraints:

Amount of C used to produce A and B: $\quad C = 0.4A + 0.5B$ (9)

Amount of D used to produce A and B: $\quad D = 0.6A + 0.5B$ (10)

Non-negativity of Design Variable:
$$A \ge 0, B \ge 0, C \ge 0, D \ge 0 \tag{11}$$

2.7———

Design a diet of bread and milk to get at least 5 units of vitamin A and 4 units of vitamin B each day. The amount of vitamins A and B in 1 kg of each food and the cost per kilogram of food are given in Table E2.7. Formulate the design optimization problem so that we get at least the basic requirements of vitamins at the minimum cost.

Table E2.7 Data for the Diet Problem

Vitamin	Bread	Milk
A	1	2
B	3	2
Cost/kg	2	1

Solution

Given: The minimum amount of vitamins A and B required each day, the amount of vitamins A and B present in one kilogram of bread and milk, and the cost per kilogram of food.
Required: It is desired to find the amount of each food which should be consumed to provide the basic vitamin requirements at the minimum cost.
Procedure: We follow the five step process to formulate the problem as an optimization problem.

Step 1: Problem Statement
    Shown above

Step 2: Data and Information Collection
    Shown above

Step 3: Definition of Design Variables
    B = bread consumed in kg
    M = milk consumed in kg

Step 4: Optimization Criterion
    Optimization criterion is to minimize the cost, and the cost function is defined as
    $Cost = 2B + M$, $

Step 5: Formulation of Constraints
    Vitamin A Constraint: $B + 2M \geq 5$
    Vitamin B Constraint: $3B + 2M \geq 4$
    Explicit Design Variable Constraints:
        $B \geq 0$, kg; $M \geq 0$, kg

2.8———————————————————————————————————

Enterprising engineering students have set up a still in a bathtub. They can produce 225 bottles of pure alcohol each week. They bottle two products from alcohol: (i) wine, 20 proof, and (ii) whiskey, 80 proof. Recall that pure alcohol is 200 proof. They have an unlimited supply of water but can only obtain 800 empty bottles per week because of stiff competition. The weekly supply of sugar is enough for either 600 bottles of wine or 1200 bottles of whiskey. They make $1.00 profit on each bottle of wine and $2.00 profit on each bottle of whiskey. They can sell whatever they produce. How many bottles of wine and whisky should they produce each week to maximize profit? Formulate the design optimization problem. (created by D. Levy)

Solution

Given: The amount of bottles of pure alcohol which can be produced each week, the two types of alcohol which are produced, the amount of empty bottles available per week, the amount of each alcohol which can be produced based on the weekly sugar supply, and the profits for each alcohol type.

Required: It is desired to find the amount of bottles of wine and whisky which should be produced, each week, to maximize profit.

Procedure: We follow the five step process to formulate the problem as an optimization problem.

Step 1: Problem Statement
　　Shown above

Step 2: Data and Information Collection
　　Shown above

Step 3: Definition of Design Variables
　　x_1 = bottles of wine produced/week
　　x_2 = bottles of whiskey produced/week

Step 4: Optimization Criterion
　　Optimization criterion is to maximize profit, and the cost function is defined as
　　$Profit = x_1 + 2x_2$

Step 5: Formulation of Constraints
　　Supply of Bottles Constraint: $x_1 + x_2 \leq 800$
　　Supply of Alcohol Constraint: $0.1x_1 + 0.4x_2 \leq 225$
　　Sugar Limitation Constraint: $x_1/600 + x_2/1200 \leq 1$
　　Explicit Design Variable Constraints:
　　　　$x_1 \geq 0, \ x_2 \geq 0$

2.9

Design a can closed at one end using the smallest area of sheet metal for a specified interior volume of 600 cm^3. The can is a right circular cylinder with interior height h and radius r. The ratio of height to diameter must not be less than 1.0 nor greater than 1.5. The height cannot be more than 20 cm. Formulate the design optimization problem.

Solution

Given: The desired interior can volume, the minimum and maximum ratio of height to diameter, and the maximum height.
Required: It is desired to find the design which minimizes the area of sheet metal for the can.
Procedure: We follow the five step process to formulate the problem as an optimization problem.

Step 1: Problem Statement
    Shown above

Step 2: Data and Information Collection
    Shown above

Step 3: Definition of Design Variables
    h = interior height of the can in cm
    r = interior radius of the can in cm

Step 4: Optimization Criterion
    Optimization criterion is to minimize area of sheet metal, and the cost function is defined as
    $Area = \pi r^2 + 2\pi rh$, , cm²

Step 5: Formulation of Constraints
    Volume Constraint: $\pi r^2 h = 600$, , cm³
    Height/Diameter Constraints:
            $h/2r \geq 1$
            $h/2r \leq 1.5$
    Explicit Design Variable Constraints:
            $h \leq 20$, cm; $h \geq 0$, cm; $r \geq 0$, cm

2.10——

Design a shipping container closed at both ends with dimensions $b \times b \times h$ to minimize the ratio: (round-trip cost of shipping the container only)/(one-way cost of shipping the contents only). Use the data in the following table. Formulate the design optimization problem.

Mass of the container/surface area	80 kg/ m^2
Maximum b	10 m
Maximum h	18 m
One-way shipping cost, full or empty	$18/kg gross mass
Mass of the contents	150 kg/ m^3

Solution

Given: The mass of the container per unit area, the maximum height and square base length of the container, the one way shipping cost, the mass of the contents, and the ratio of a round trip cost of shipping the container to a one way cost of shipping its contents only.

Required: It is desired to find the design of the shipping container which minimizes the ratio given.

Procedure: We follow the five step process to formulate the problem as an optimization problem.

Step 1: Problem Statement
    Shown above

Step 2: Data and Information Collection
    Shown above

Step 3: Definition of Design Variables
    b = base of the container, m
    h = height of the container, m

Step 4: Optimization Criterion
    Optimization criterion is to minimize a ratio, and the cost function is defined as

$$Ratio = \frac{\text{round - trip cost of shipping the container}}{\text{one - way cost of shipping the contents}}$$

$$= \frac{2(18)(80)\left(2b^2 + 4bh\right)}{(18)(150)\left(b^2\, h\right)} = \left(\frac{32}{15}\right)\left(\frac{b^2 + 2bh}{b^2\, h}\right) = \left(\frac{32}{15}\right)\left(\frac{1}{h} + \frac{2}{b}\right)$$

Step 5: Formulation of Constraints
    Explicit Design Variable Constraints:
        $b \leq 10$, m
        $h \leq 18$, m
        $b \geq 0$, m
        $h \geq 0$, m

2.11

Certain mining operations require an open top rectangular container to transport materials. The data for the problem are as follows:

Construction costs:
- Sides: $50/m^2
- Ends: $60/m^2
- Bottom: $90/m^2

Minimum volume needed: 150 m^3

Formulate the problem of determining the container dimensions for minimum present cost.

Solution

Given: The construction costs for the sides, ends, and the bottom of the container and the minimum volume requirement.

Required: It is desired to find the dimensions of the material container which minimize cost.

Procedure: We follow the five step process to formulate the problem as an optimization problem.

Step 1: Problem Statement
   Shown above

Step 2: Data and Information Collection
   Shown above

Step 3: Definition of Design Variables
   dimensions of the container;
   b = width, m
   d = depth, m
   h = height, m

Step 4: Optimization Criterion
   Optimization criterion is to minimize total present cost, and the cost function is defined as
   $$Cost = \left[2dh(50) + 2bh(60) + bd(90) \right]$$
   $$Cost = (100dh + 120bh + 90bd)$$

Step 5: Formulation of Constraints
   Volume Constraint:
   $$bdh \geq 150, \text{ m}^3$$

   Explicit Design Variable Constraints:
   $$b \geq 0, \text{ m}; \quad d \geq 0, \text{ m}; \quad h \geq 0, \text{ m}$$

2.12————————————————————————————————

Design a circular tank closed at both ends to have a volume of 250 m³. The fabrication cost is proportional to the surface area of the sheet metal and is \$400/m². The tank is to be housed in a shed with a sloping roof. Therefore, height H of the tank is limited by the relation $H \leq (10 - D/2)$, where D is the tank's diameter. Formulate the minimum-cost design problem.

Solution

Given: The required volume of the tank, the fabrication cost of the sheet metal per unit area, and the limiting relation between the height and the diameter.

Required: It is desired to find a design of the tank which minimizes cost.

Procedure: We follow the five step process to formulate the problem as an optimization problem.

Step 1: Problem Statement
   Shown above

Step 2: Data and Information Collection
   Shown above

Step 3: Definition of Design Variables
   D = diameter of the tank in m
   H = height of the tank in m

Step 4: Optimization Criterion
   Optimization criterion is to minimize the cost, and the cost function is defined as
   $Cost = 400(\pi D^2/2 + \pi DH)$

Step 5: Formulation of Constraints
   Constraint: $\pi D^2 H/4 = 250$, m³
   Constraint: $H \leq 10 - D/2$, m
   Explicit Design Variable Constraints:
       $H \geq 0$, m; $D \geq 0$, m

2.13——

Design the steel framework shown in Figure E2.13 at a minimum cost. The cost of a horizontal member in one direction is \$20w and in the other direction it is \$30d. The cost of a vertical column is \$50h. The frame must enclose a total volume of at least 600 m^3. Formulate the design optimization problem.
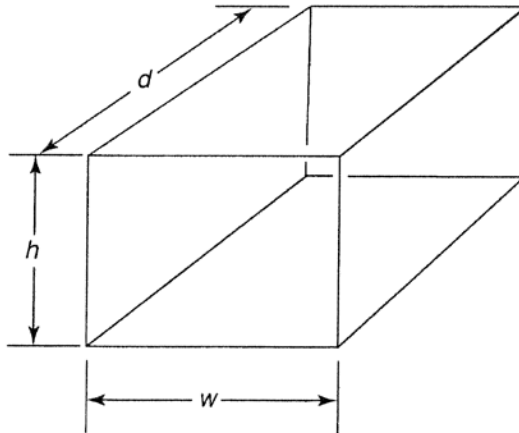

FIGURE E2.13 Steel frame.

Solution

Given: The cost of a horizontal member in two, separate directions, the cost of a vertical member, and the minimum volume which must be enclosed.
Required: It is desired to find a design which minimizes the cost of the steel framework.
Procedure: We follow the five step process to formulate the problem as an optimization problem.

Step 1: Problem Statement
    Shown above

Step 2: Data and Information Collection
    Shown above

Step 3: Definition of Design Variables
    w = width of the frame, m
    d = depth of the frame, m
    h = height of the frame, m

Step 4: Optimization Criterion
    Optimization criterion is to minimize the cost, and the cost function is defined as
    $Cost = 80w + 120d + 200h$

Step 5: Formulation of Constraints
    Volume Constraint: $wdh \geq 600$, m^3
    Explicit Design Variable Constraints:
        $w, d, h \geq 0$, m

2.14———————————————————————————————————

Two electric generators are interconnected to provide total power to meet the load. Each generator's cost is a function of the power output, as shown in Figure E2.14. All costs and power are expressed on a per unit basis. The total power needed is at least 60 units. Formulate a minimum-cost design problem to determine the power outputs P_1 and P_2.
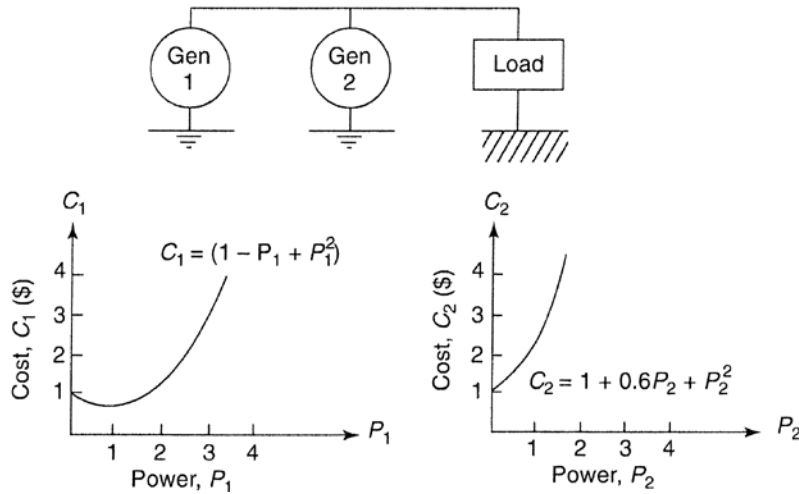

FIGURE E2.14 Power generator.

Solution

Given: The cost function of each generator, shown in Figure E2.14, and the minimum total power needed.

Required: It is desired to find the power outputs, P_1 and P_2, which minimizes cost.

Procedure: We follow the five step process to formulate the problem as an optimization problem.

Step 1: Problem Statement
    Shown above

Step 2: Data and Information Collection
    Shown above

Step 3: Definition of Design Variables
    P_1 = Number of power units for generator one
    P_2 = Number of power units for generator two

Step 4: Optimization Criterion
    Optimization criterion is to minimize the cost, and the cost function is defined as
    $$Cost = C_1 + C_2 = (1 - P_1 + P_1^2) + (1 + 0.6P_2 + P_2^2)$$

Step 5: Formulation of Constraints
    Constraint: $P_1 + P_2 \geq 60$
    Explicit Design Variable Constraints:
        $P_1 \geq 0; \quad P_2 \geq 0$

2.15

Transportation Problem. A company has m manufacturing facilities. The facility at the ith location has capacity to produce b_i units of an item. The product should be shipped to n distribution centers. The distribution center at the jth location requires at least a_j units of the item to satisfy demand. The cost of shipping an item from the ith plant to the jth distribution center is c_{ij}. Formulate a minimum-cost transportation system to meet each distribution center's demand without exceeding the capacity of any manufacturing facility.

<u>Solution</u>

Given: The number of manufacturing facilities the company owns, the capacity of the ith facility to produce b_i units of an item, the number of distribution centers the product should be shipped too, the minimum number of items, a_j, required by the jth distribution center, and the cost to ship an item from the ith plant to the jth distribution center.

Required: It is desired to design a transportation system which minimizes costs and meets the constraints set by the two types of facilities.

Procedure: We follow the five step process to formulate the problem as an optimization problem.

Step 1: Problem Statement
　　Shown above

Step 2: Data and Information Collection
　　Shown above

Step 3: Definition of Design Variables
　　x_{ij} : number of items produced at the ith facility shipped to jth distribution center
　　where $i = 1$ to m; $j = 1$ to n

Step 4: Optimization Criterion
　　Optimization criterion is to minimize the cost, and the cost function is defined as

$$Cost = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Step 5: Formulation of Constraints
　　Capacity of Manufacturing Facility Constraint: $\sum_{j=1}^{n} x_{ij} \leq b_i$ for $i = 1$ to m

　　Demand Constraint: $\sum_{i=1}^{m} x_{ij} \geq a_j$ for $j = 1$ to n; $\quad x_{ij} \geq 0$ for all i and j

2.16———————————————————————————————————————

Design of a two-bar truss. Design a symmetric two-bar truss (both members have the same cross section), as shown in Fig. E2.16, to support a load W. The truss consists of two steel tubes pinned together at one end and supported on the ground at the other. The span of the truss is fixed at s. Formulate the minimum mass truss design problem using height and the cross-sectional dimensions as design variable. The design should satisfy the following constraints:

1. Because of space limitations, the height of the truss must not exceed b_1, and must not be less than b_2.
2. The ratio of the mean diameter to thickness of the tube must not exceed b_3.
3. The compressive stress in the tubes must not exceed the allowable stress, σ_a, for steel.
4. The height, diameter, and thickness must be chosen to safeguard against member buckling.

Use the following data: $W = 10$ kN; span $s = 2$ m; $b_1 = 5$ m; $b_2 = 2$ m; $b_3 = 90$; allowable stress, σ_a =250 MPa; modulus of elasticity, $E = 210$ GPa; mass density, ρ =7850 kg/m^3; factor of safety against buckling; FS=2; $0.1 \leq D \leq 2$, m) and $0.01 \leq t \leq 0.1$, m.
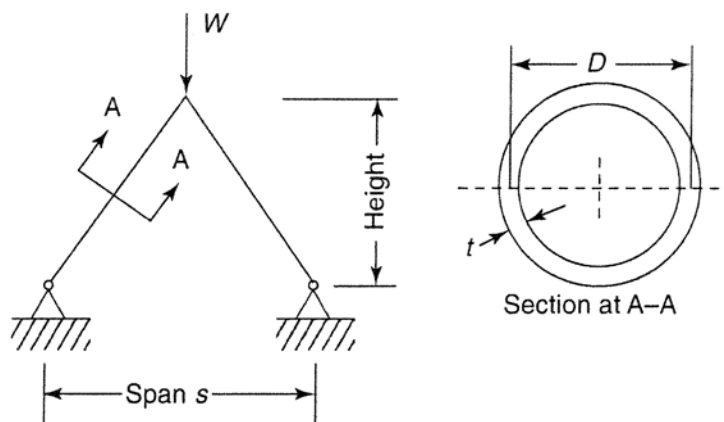

FIGURE E2.16 Two-bar structure.

Solution

Given: Constraints 1-4 listed above and the factor of safety against buckling in the data section above.

Required: It is desired to design a truss which minimizes mass using height and the cross sectional dimensions as design variables.

Procedure: We follow the five step process to formulate the problem as an optimization problem.

Step 1: Problem Statement
Shown above

Step 2: Data and Information Collection
Depending on the units used for various parameters, the final expressions for various function will look different. The following table give values of various parameters depending on the units used:

Variable	N & m	N & mm	N & cm	KN & m	MN & m
Load, W	10,000	10,000	10,000	10	1×10^{-2}
σ_a	250×10^6	250	250×10^2	250×10^3	250
Modulus, E	210×10^9	210×10^3	210×10^5	210×10^6	210×10^3
Density, ρ	7850	7.85×10^{-6}	7.85×10^{-3}	7850	7850
Span, s	2	2000	200	2	2
b_1	5	5000	500	5	5
b_2	2	2000	200	2	2
D_{min}	0.1	100	10	0.1	0.10
D_{max}	2	2000	200	2	2
t_{min}	0.01	10	1	0.01	0.01
t_{max}	0.1	100	10	0.1	0.1

Other data/expressions that need to be collected are:

Member length, $l = \sqrt{H^2 + (0.5s)^2}$

Member force: Draw the free-body diagram of the loaded node and sum up the forces in the vertical direction:

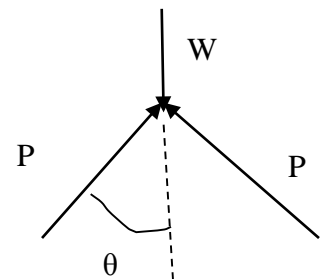$-W + 2P\cos\theta = 0;\quad or \quad P = \frac{W}{2\cos\theta};\quad \cos\theta = \frac{H}{l}$

Member stress: $\sigma = \frac{P}{A}$

Cross-sectional area: The expression will depend on what variables are used:

$$A = \frac{\pi}{4}(D_o^2 - D_i^2) = \pi Dt$$

Moment of inertia: $I = \frac{\pi}{64}(D_o^4 - D_i^4) = \frac{\pi}{8}(D^3 t + D t^3)$

Buckling load (critical load) for pin-pin column: $P_{cr} = \frac{\pi^2 EI}{l^2}$

FORMULATION 1: In terms of intermediate variables

Step 3: Definition of Design Variables
H = height of the truss, m
D = mean diameter of the tube, m
t = thickness of the tube, m

Step 4: Optimization Criterion
Optimization criterion is to minimize mass, and the cost function is defined as

$Mass = 2\rho Al$
where ρ is the mass density of the material.

Step 5: Formulation of Constraints

Stress Constraint: $\qquad\qquad \sigma \leq \sigma_a$

Buckling Constraint: $\qquad\quad P \leq \frac{P_{cr}}{FS}$

Explicit Design Variable Constraints:

$H \leq b_1; \ H \geq b_2; \ D/t \leq b_3;$
$0.1 \leq D \leq 2$ m; $\quad 0.1 \leq t \leq 0.1$ m

FORMULATION 2: Explicitly in terms of the design variables.

Use N and m as the units, and the corresponding values for various parameters.

Member Force: $P = W(s^2/4 + H^2)^{\frac{1}{2}}/2H$

Step 3: Definition of Design Variables
H = height of the truss, m
D = mean diameter of the tube, m
t = thickness of the tube, m

Step 4: Optimization Criterion
Optimization criterion is to minimize mass, and the cost function is defined as

$Mass = 2\rho Al = 2\rho(\pi Dt)\left(s^2/4 + H^2\right)^{\frac{1}{2}}$;

where ρ is the mass density of the material.

Substituting the given values, we get

$Mass = 2(7850)(\pi Dt)(1 + H^2)^{\frac{1}{2}} = 49323\, Dt(1 + H^2)^{\frac{1}{2}}$, kg

Step 5: Formulation of Constraints
Stress Constraint: $P/A \leq \sigma_a$; $W(s^2/4 + H^2)^{\frac{1}{2}}/2H\,(\pi Dt) \leq \sigma_a$

Buckling Constraint: $P \leq P_{cr}/(FS)$; $P_{cr} = \pi^2 EI/l^2 = \dfrac{\pi^2 E\left[\pi\left(D^3t + Dt^3\right)/8\right]}{\left(s^2/4 + H^2\right)}$

Or, $\dfrac{W\left(s^2/4 + H^2\right)^{\frac{1}{2}}}{2H} \leq \dfrac{\pi^2 E\left[\pi\left(D^3t + Dt^3\right)/8\right]}{(FS)\left(s^2/4 + H^2\right)}$

Explicit Design Variable Constraints: $H \leq b_1$; $H \geq b_2$; $D/t \leq b_3$;
$0.1 \leq D \leq 2$ m; $0.1 \leq t \leq 0.1$ m

Substituting the given data, we obtain the final form of the constraints as

$10000(1 + H^2)^{\frac{1}{2}}/2\pi HDt \leq 250 \times 10^6$
$10000(1 + H^2)^{\frac{1}{2}}/2H \leq \left(210 \times 10^9\right)\pi^3\left(D^3t + Dt^3\right)/16\left(1 + H^2\right)$
$H \leq 5$, m;
$H \geq 2$, m;
$D/t \leq 90$;
$0.1 \leq D \leq 2$, m;
$0.01 \leq t \leq 0.1$, m
$H \leq 5$, m;

2.17───

A beam of rectangular cross section (Fig. E2.17) is subjected to a maximum bending moment of M and maximum shear of V. The allowable bending and shearing stresses are σ_a and τ_a, respectively. The bending stress in the beam is calculated as

$$\sigma = \frac{6M}{bd^2}$$

and average shear stress in the beam is calculated as

$$\tau = \frac{3V}{2bd}$$

where d is the depth and b is the width of the beam. It is also desired that the depth of the beam shall not exceed twice its width. Formulate the design problem for minimum cross-sectional area using this data: $M=140$ kN·m, $V=24$ kN, $\sigma_a=165$ MPa, $\tau_a=50$ MPa.
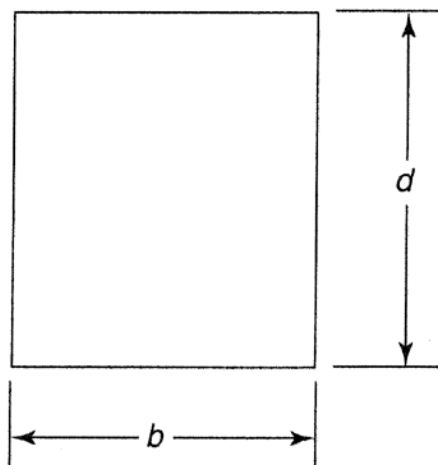

FIGURE E2.17 Cross section of a rectangular beam.

<u>**Solution**</u>

Given: The equations to calculate bending and average shear stress in a beam, the constraint that the depth of the beam will not exceed twice its width, the applied moment, the applied shear force, and the maximum allowable bending and shear stresses in the beam.

Required: It is desired to design a beam which minimizes cross-sectional area without yielding due to shear or bending stresses.

Procedure: We follow the five step process to formulate the problem as an optimization problem.

Step 1: Problem Statement
  Shown above

Step 2: Data and Information Collection
  $M = 140$ kN.m $= 1.4 \times 10^7$ N.cm;
  $V = 24$ kN $= 2.4 \times 10^4$ N;

───

$\sigma_a = 165$ MPa $= 1.65 \times 10^4$ N/cm^2;

$t_a = 50$ MPa $= 5000$ N/cm^2

Step 3: Definition of Design Variables
b = width of the beam, cm
d = depth of the beam, cm

Step 4: Optimization Criterion
Optimization criterion is to minimize the cross-sectional area, and the cost function is defined as
$Area = bd$, cm^2

Step 5: Formulation of Constraints
Bending Stress Constraint: $6M/bd^2 \leq \sigma_a$ or $6(1.4 \times 10^7)/bd^2 \leq 1.65 \times 10^4$

Shear Stress Constraint: $3V/2bd \leq t_a$ or $3(2.4 \times 10^4)/2bd \leq 5000$

Constraint: d $\leq$ 2b or $d - 2b \leq 0$
Explicit Design Variable Constraints: b, d $\geq$ 0

From the graph for the problem, we get the optimum solution as
$b^* \doteq 10.8$ cm, $d^* \doteq 21.6$ cm, $Area \doteq 233$ cm^2 where constraint numbers 1 and 3 are active.

2.18——————————————————————————————

A vegetable oil processor wishes to determine how much shortening, salad oil, and margarine to produce to optimize the use of his current oil stock supply. At the present time, he has 250,000 kg of soybean oil, 110,000 kg of cottonseed oil, and 2000 kg of milk-base substances. The milk-base substances are required only in the production of margarine. There are certain processing losses associated with each product: 10 percent for shortening, 5 percent for salad oil, and no loss for margarine. The producer's back orders require him to produce at least 100,000 kg of shortening, 50,000 kg of salad oil, and 10,000 kg of margarine. In addition, sales forecasts indicate a strong demand for all produces in the near future. The profit per kilogram and the base stock required per kilogram of each product are given in Table E2.18. Formulate the problem to maximize profit over the next production scheduling period. (created by J. Liittschwager).

Table E2.18 Data for the Vegetable Oil Processing Problem

		Parts per kg of base stock Requirements		
Product	Profit per kg	Soybean	Cottonseed	Milk base
Shortening	1.0	2	1	0
Salad oil	0.8	0	1	0
Margarine	0.5	3	1	1

Solution

Given: The current supply of soybean oil, cottonseed oil, and milk-base substances, milk-base substances are required in the production of margarine only, the amount of processing loss which occurs in shortening, salad oil, and margarine, the minimum production requirement of each product, and the data shown in Table E2.18.
Required: It is desired to create a production schedule which will maximize profit.
Procedure: We follow the five step process to formulate the problem as an optimization problem.

Step 1: Problem Statement
Shown above

Step 2: Data and Information Collection
Shown above

Step 3: Definition of Design Variables
x_1 = shortening produced after losses, kg
x_2 = salad oil produced after losses, kg
x_3 = margarine produced, kg

Step 4: Optimization Criterion
Optimization criterion is to maximize the profit, and the cost function is defined as
$Profit = x_1 + 0.8\,x_2 + 0.5\,x_3$

Step 5: Formulation of Constraints
The ingredients used cannot exceed current stocks

Soybean Constraint: $(2x_1/3)(1/0.9) + (3x_3/5) \leq 250{,}000$

Milk Base Constraint: $(x_3/5) \leq 2000$

Cottonseed Constraint: $(x_1/3)(1/0.9) + (x_2)(1/0.95) + (x_3/5) \leq 110{,}000$

The demand for the needs of the products to be satisfied

Explicit Design Variable Constraints: $x_1 \geq 100{,}000; \quad x_2 \geq 50{,}000; \quad x_3 \geq 10{,}000$

Section 2.11   A General Mathematical Model for Optimum Design

2.19

Answer True or False.

1. Design of a system implies specification for the design variable values. *True*
2. All design problems have only linear inequality constraints. *False*
3. All design variables should be independent of each other as far as possible. *True*
4. If there is an equality constraint in the design problem, the optimum solution must satisfy it. *True*
5. Each optimization problem must have certain parameters called the design variables. *True*
6. A feasible design may violate equality constraints. *False*
7. A feasible design may violate "≥type' constraints. *False*
8. A "≤ type" constraint expressed in the standard form is active at a design point if it has zero value there. *True*
9. The constraint set for a design problem consists of all the feasible points. *True*
10. The number of independent equality constraints can be larger than the number of design variables for the problem. *True*
11. The number of "≤ type" constraints must be less than the number of design variables for a valid problem formulation. *False*
12. The feasible region for an equality constraint is a subset of that for the same constraint expressed as an inequality. *True*
13. Maximization of $f(x)$ is equivalent to minimization of $1/f(x)$. *False*
14. A lower minimum value for the cost function is obtained if more constraints are added to the problem formulation. *False*
15. Let f_n be the minimum value for the cost function with *n* design variables for a problem. If the number of design variables for the same problem is increased to, say $m = 2n$, then $f_m > f_n$ where f_m is the minimum value for the cost function with *m* design variables. *False*

2.20*——

A trucking company wants to purchase several new trucks. It has $2 million to spend. The investment should yield a maximum of trucking capacity for each day in tonnes×kilometers. Data for the three available truck models are given in Table E2.20: i.e., truck load capacity speed, crew required/shift, hours of operations for three shifts, and the cost of each truck. There are some limitations on the operations that need to be considered. The labor market is such that the company can hire at most 150 truck drivers. Garage and maintenance facilities can handle at the most 25 trucks. How many trucks of each type should the company purchase? Formulate the design optimization problem.

Table E2.20 Data for Available Trucks

Truck model	Truck load Capacity (tones)	Average truck speed (km/h)	Crew required per shift	No. of hours of operations per day (3 shifts)	Cost of each truck($)
A	10	66	1	19	40,000
B	20	50	2	18	60,000
C	18	50	2	21	70,000

Solution

Given: The maximum amount of money the company can spend, the data given in Table E2.20, the maximum number of truck drivers which can be hired, and the maximum number of trucks which can be purchased.

Required: It is desired to purchase the appropriate number of each truck which will satisfy the constraints and maximize yield of trucking capacity each day in tonnes×kilometers.

Procedure: We follow the five step process to formulate the problem as an optimization problem.

Step 1: Problem Statement
Shown above

Step 2: Data and Information Collection
Shown above

Step 3: Definition of Design Variables
A, B and C are the number of trucks to be purchased of the type A, B and C, respectively.

Step 4: Optimization Criterion
Optimization criterion is to maximize the capacity (tonnes×kilometers), and the cost function is defined as:

$Capacity = A(10 \times 55 \times 18) + B(20 \times 50 \times 18) + C(18 \times 50 \times 21) = 9900A + 18000B + 18900C$

Transcribing into the standard form, we get:

minimize $f = -9900A - 18000B - 18900C$

Step 5: Formulation of Constraints

Available Capital Constraint: $A(40,000) + B(60,000) + C(70,000) \leq 2,000,000$
Limit on Available Drivers Constraint: $3A + 6B + 6C \leq 150$
Limitation on Maintenance Facility Constraint: $A + B + C \leq 30$
Explicit Design Variable Constraints: $A, B, C \geq 0$

Transcribing into the standard form, we get:

$g_1 = (40000A + 60000B + 70000C) - 2000000 \leq 0$:
$g_2 = (3A + 6B + 6C) - 150 \leq 0$;
$g_3 = (A + B + C) - 30 \leq 0$;
$-A \leq 0$
$-B \leq 0$
$-C \leq 0$

2.21*——

A large steel corporation has two iron ore reduction plants. Each plant processes iron ore into two different ingot stocks. They are shipped to any of the three fabricating plants where they are made into either of the two finished products. In total, there are two reduction plants, two ingot stocks, three fabricating plants, and two finished products.

For the coming season, the company wants to minimize total tonnage of iron ore processed in its reduction plants, subject to production and demand constraints. Formulate the design optimization problem and transcribe it into the standard model.

Nomenclature

$a(r, s)$=tonnage yield of ingot stock s from 1 ton of iron ore processed at reduction plant r
$b(s, f, p)$= total yield from 1 ton of ingot stock s shipped to fabricating plant f and manufactured into product p
$c(r)$=iron ore processing capacity in tonnage at reduction plant r
$k(f)$=capacity of the fabricating plant f in tonnage for all stocks
$D(p)$=tonnage demand requirement for product p

Production and demand constraints

1. The total tonnage of iron ore processed by both reduction plants must equal the total tonnage processed into ingot stocks for shipment to the fabricating plants.
2. The total tonnage of iron ore processed by each reduction plant cannot exceed its capacity.
3. The total tonnage of ingot stock manufactured into products at each fabricating plant must equal the tonnage of ingot stock shipped to it by the reduction plants.
4. The total tonnage of ingot stock manufactured into products at each fabrication plant cannot exceed its available capacity.
5. The total tonnage of each product must equal its demand.

Constants for the problem

$a(1,1)$=0.39	$c(1)$=1,200,000	$k(1)$=190,000	$D(1)$=330,000
$a(1,2)$=0.46	$c(2)$=1,000,0 00	$k(2)$=240,000	$D(2)$=125,000
$a(2,1)$=0.44		$k(3)$=290,000	
$a(2,2)$=0.48			
		$b(1,1,1)$=0.79	$b(1,1,2)$=0.84
		$b(2,1,1)$=0.68	$b(2,1,2)$=0.81
		$b(1,2,1)$=0.73	$b(1,2,2)$=0.85
		$b(2,2,1)$=0.67	$b(2,2,2)$=0.77
		$b(1,3,1)$=0.74	$b(1,3,2)$=0.72
		$b(2,3,1)$=0.62	$b(2,3,2)$=0.78

Solution

Given: The maximum number of reduction plants, ingot stocks, fabricating plants, and finished products available, the constraints 1-5 shown above, and the constants shown in the table above.

Required: It is desired to minimize the total tonnage of iron ore that is processed in reduction plants.

Procedure: We follow the five step process to formulate the problem as an optimization problem.

Several formulations for the design problem are possible. For each formulation proper design variables are identified. Expressions for the cost and constraint functions are derived.

Formulation 1:

Step 1: Problem Statement
Shown above

Step 2: Data and Information Collection
Shown above

Step 3: Definition of Design Variables
For this formulation, twenty-four design variables are chosen which designate the twenty-four different paths of processing the iron ore, i.e., R(i, j, k, l) with i=1, 2; j=1, 2; k=1, 2, 3; and l =1, 2. For a particular set of i, j, k, and l, R(i, j, k, l) means that the tonnage of iron ore processed at reduction plant i, yielding ingot stock j, shipped to the fabricating plant k and manufactured into product l.

For simplicity of the following derivation, let

x_1 = R(1,1,1,1); x_2 = R(1,1,1,2); x_3 = R(1,2,1,1); x_4 = R(1,2,1,2); x_5 = R(1,1,2,1); x_6 = R(1,1,2,2);
x_7 = R(1,2,2,1); x_8 = R(1,2,2,2); x_9 = R(1,1,3,1); x_{10} = R(1,1,3,2); x_{11}= R(1,2,3,1); x_{12}= R(1,2,3,2);
y_1 = R(2,1,1,1); y_2 = R(2,1,1,2); y_3 = R(2,2,1,1); y_4 = R(2,2,1,2); y_5 = R(2,1,2,1); y_6 = R(2,1,2,2);
y_7 = R(2,2,2,1); y_8 = R(2,2,2,2); y_9 = R(2,1,3,1); y_{10} = R(2,1,3,2); y_{11}= R(2,2,3,1); y_{12}= R(2,2,3,2)

Step 4: Optimization Criterion
Optimization criterion is to minimize total tonnage of iron ore processed at the reduction plants, and the cost function is defined as

$$f = \sum_{i=1}^{12} x_i + \sum_{j=1}^{12} y_i$$

Summarizing and transcribing into the standard model, we get

$$f = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} + y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7$$
$$+ y_8 + y_9 + y_{10} + y_{11} + y_{12}$$

Step 5: Formulation of Constraints

(1) The total tonnage of iron ore processed by each reduction plant cannot exceed its capacity, i.e., RP1 $\leq c(1)$; RP2 $\leq c(2)$ where RP1 and RP2 represent the total tonnage of iron ore processed at the two reduction plants separately. In terms of design variables and the given data, these two constraints are:

$$g_1: \sum_{i=1}^{12} x_i \leq 1,200,000; \qquad g_2: \sum_{j=1}^{12} y_j \leq 1,000,000$$

(2) The total tonnage of ingot stocks manufactured into products at each fabricating plant cannot exceed its available capacity, i.e., $F_1 \leq k(1)$; $F_2 \leq k(2)$; $F_3 \leq k(3)$ where F_1, F_2 and F_3 represent the total tonnage of ingot stocks processed at three fabricating plants separately. In terms of design variables and the given data, these constraints can be written as:

g_3: $a(1,1)(x_1 + x_2) + a(1,2)(x_3 + x_4) + a(2,1)(y_1 + y_2) + a(2,2)(y_3 + y_4) \leq 190,000$; or
$\quad$: $0.39(x_1 + x_2) + 0.46(x_3 + x_4) + 0.44(y_1 + y_2) + 0.48(y_3 + y_4) \leq 190,000$
g_4: $0.39(x_5 + x_6) + 0.46(x_7 + x_8) + 0.44(y_5 + y_6) + 0.48(y_7 + y_8) \leq 240,000$
g_5: $0.39(x_9 + x_{10}) + 0.46(x_{11} + x_{12}) + 0.44(y_9 + y_{10}) + 0.48(y_{11} + y_{12}) \leq 290,000$

(3) The total tonnage of each product p_1 and p_2 respectively, must be equal to its demand, i.e., $p_1 = D(1)$; $p_2 = D(2)$ In terms of the design variables and the given data, the two constraints are written as:

$$h_1: \sum_{i=1,3,5,7,9,11} (e_i x_i + f_i y_i) = 330,000; \qquad h_2: \sum_{i=2,4,6,8,10,12} (e_i x_i + f_i y_i) = 125,000$$

where e_i's and f_i's are coefficients transferring tonnage of iron ore into products. These coefficients are given as:

$e_1 = a(1,1)\,b(1,1,1) = 0.39(0.79) = 0.3081$; $\quad e_2 = a(1,1)\,b(1,1,2) = 0.39(0.84) = 0.3276$
$e_3 = a(1,2)\,b(2,1,1) = 0.46(0.68) = 0.3128$; $\quad e_4 = a(1,2)\,b(2,1,2) = 0.46(0.81) = 0.3726$
$e_5 = a(1,1)\,b(1,2,1) = 0.39(0.73) = 0.2847$; $\quad e_6 = a(1,1)\,b(1,2,2) = 0.39(0.85) = 0.3315$
$e_7 = a(1,2)\,b(2,2,1) = 0.46(0.67) = 0.3082$; $\quad e_8 = a(1,2)\,b(2,2,2) = 0.46(0.77) = 0.3542$
$e_9 = a(1,1)\,b(1,3,1) = 0.39(0.74) = 0.2886$; $\quad e_{10} = a(1,1)\,b(1,3,2) = 0.39(0.72) = 0.2808$
$e_{11} = a(1,2)\,b(2,3,1) = 0.46(0.62) = 0.2852$; $\quad e_{12} = a(1,2)\,b(2,3,2) = 0.46(0.78) = 0.3588$
$f_1 = a(2,1)\,b(1,1,1) = 0.44(0.79) = 0.3476$; $\quad f_2 = a(2,1)\,b(1,1,2) = 0.44(0.84) = 0.3696$
$f_3 = a(2,2)\,b(2,1,1) = 0.48(0.68) = 0.3264$; $\quad f_4 = a(2,2)\,b(2,1,2) = 0.48(0.81) = 0.3888$
$f_5 = a(2,1)\,b(1,2,1) = 0.44(0.73) = 0.3212$; $\quad f_6 = a(2,1)\,b(1,2,2) = 0.44(0.85) = 0.3740$
$f_7 = a(2,2)\,b(2,2,1) = 0.48(0.67) = 0.3216$; $\quad f_8 = a(2,2)\,b(2,2,2) = 0.48(0.77) = 0.3696$
$f_9 = a(2,1)\,b(1,3,1) = 0.44(0.74) = 0.3256$; $\quad f_{10} = a(2,1)\,b(1,3,2) = 0.44(0.72) = 0.3168$
$f_{11} = a(2,2)\,b(2,3,1) = 0.48(0.62) = 0.2976$; $\quad f_{12} = a(2,2)\,b(2,3,2) = 0.48(0.78) = 0.3744$

(4) There are constraints requiring that both the reduction plants and fabricating plants do not have any inventory of their own. These constraints have been satisfied automatically since the twenty-four design variables (paths) are chosen which satisfy these conditions.

Summarizing and transcribing into the standard model, we get

$h_1 = 0.3081\, x_1 + 0.3128\, x_3 + 0.2847\, x_5 + 0.3082\, x_7 + 0.2886\, x_9 + 0.2852\, x_{11} + 0.3476\, y_1 + 0.3264\, y_3 + 0.3212\, y_5 + 0.3216\, y_7 + 0.3256\, y_9 + 0.2976\, y_{11} - 330{,}000 = 0$

$h_2 = 0.3276\, x_2 + 0.3726\, x_4 + 0.3315\, x_6 + 0.3542\, x_8 + 0.2808\, x_{10} + 0.3588\, x_{12} + 0.3696\, y_2 + 0.3888\, y_4 + 0.3740\, y_6 + 0.3696\, y_8 + 0.3168\, y_{10} + 0.3744\, y_{12} - 125{,}000 = 0$

$g_1 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} - 1{,}200{,}000 \le 0$

$g_2 = y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10} + y_{11} + y_{12} - 1{,}000{,}000 \le 0$

$g_3 = 0.39\, x_1 + 0.39\, x_2 + 0.46\, x_3 + 0.46\, x_4 + 0.44\, y_1 + 0.44\, y_2 + 0.48\, y_3 + 0.48\, y_4 - 190{,}000 \le 0$

$g_4 = 0.39\, x_5 + 0.39\, x_6 + 0.46\, x_7 + 0.46\, x_8 + 0.44\, y_5 + 0.44\, y_6 + 0.48\, y_7 + 0.48\, y_8 - 240{,}000 \le 0$

$g_5 = 0.39\, x_9 + 0.39\, x_{10} + 0.46\, x_{11} + 0.46\, x_{12} + 0.44\, y_9 + 0.44\, y_{10} + 0.48\, y_{11} + 0.48\, y_{12} - 290{,}000 \le 0$

$-x_i \le 0, \;\; -y_i \le 0, \;\; i = 1 \text{ to } 12$

Formulation 2:

Step 1: Problem Statement
Shown above

Step 2: Data and Information Collection
Shown above

Step 3: Definition of Design Variables
The design variables are chosen as follows:

x_1 : total tonnage of iron ore processed by plant 1;
x_2 : total tonnage of iron ore processed by plant 2
x_3 : tonnage of ore processed by plant 1 for ingot stock 1;
x_4 : tonnage of ore processed by plant 1 for ingot stock 2
x_5 : tonnage of ore processed by plant 2 for ingot stock 1;
x_6 : tonnage of ore processed by plant 2 for ingot stock 2
x_7 : total tonnage yield of ingot stock 1;
x_8 : total tonnage yield of ingot stock 2
x_9 : tonnage of ingot stock 1 shipped to fabricating plant 1 to yield product 1
x_{10} : tonnage of ingot stock 1 shipped to fabricating plant 1 to yield product 2
x_{11} : tonnage of ingot stock 1 shipped to fabricating plant 2 to yield product 1
x_{12} : tonnage of ingot stock 1 shipped to fabricating plant 2 to yield product 2
x_{13} : tonnage of ingot stock 1 shipped to fabricating plant 3 to yield product 1
x_{14} : tonnage of ingot stock 1 shipped to fabricating plant 3 to yield product 2
x_{15} : tonnage of ingot stock 2 shipped to fabricating plant 1 to yield product 1
x_{16} : tonnage of ingot stock 2 shipped to fabricating plant 1 to yield product 2
x_{17} : tonnage of ingot stock 2 shipped to fabricating plant 2 to yield product 1
x_{18} : tonnage of ingot stock 2 shipped to fabricating plant 2 to yield product 2
x_{19} : tonnage of ingot stock 2 shipped to fabricating plant 3 to yield product 1
x_{20} : tonnage of ingot stock 2 shipped to fabricating plant 3 to yield product 2

Step 4: Optimization Criterion
The cost function is defined as

minimize $f = x_1 + x_2$

which is already in the standard form

Step 5: Formulation of Constraints

(1) The first constraint, total tonnage into each reduction plant must be equal to the tonnage processed into ingot stocks for shipment, implies that there will be no stock piling at the reduction plants:
$x_1 = x_3 + x_4$;
$x_2 = x_5 + x_6$

(2) The second constraint requires that the iron ore processed by each reduction plant should not exceed its maximum capacity:
$x_3 + x_4 \leq 1{,}200{,}000$;
$x_5 + x_6 \leq 1{,}000{,}000$

(3) The third constraint states that there is no stock piling at the fabricating plants. By the definition of design variables, these are:
$x_7 = 0.39\ x_3 + 0.44\ x_5$;
$x_8 = 0.46\ x_4 + 0.48\ x_6$;
$x_7 = x_9 + x_{10} + x_{11} + x_{12} + x_{13} + x_{14}$;
$x_8 = x_{15} + x_{16} + x_{17} + x_{18} + x_{19} + x_{20}$

(4) The fourth constraint is on the maximum capacity of ingot stocks at each fabricating plant:
$x_9 + x_{10} + x_{15} + x_{16} \leq 190{,}000$;
$x_{11} + x_{12} + x_{17} + x_{18} \leq 240{,}000$;
$x_{13} + x_{14} + x_{19} + x_{20} \leq 290{,}000$

(5) The fifth constraint states that the total tonnage of each product must be equal to its demand:
$0.79\ x_9 + 0.73\ x_{11} + 0.74\ x_{13} + 0.68\ x_{15} + 0.67\ x_{17} + 0.62\ x_{19} = 330{,}000$;
$0.84\ x_{10} + 0.85\ x_{12} + 0.72\ x_{14} + 0.81\ x_{16} + 0.77\ x_{18} + 0.78\ x_{20} = 125{,}000$

In the standard form, the constraints become

$h_1 = x_1 - x_3 - x_4 = 0$;
$h_2 = x_2 - x_5 - x_6 = 0$;
$h_3 = x_7 - 0.39\ x_3 - 0.44\ x_5 = 0$;
$h_4 = x_8 - 0.46\ x_4 - 0.48\ x_6 = 0$;
$h_5 = x_7 - x_9 - x_{10} - x_{11} - x_{12} - x_{13} - x_{14} = 0$;
$h_6 = x_8 - x_{15} - x_{16} - x_{17} - x_{18} - x_{19} - x_{20} = 0$;
$h_7 = 0.79\ x_9 + 0.73\ x_{11} + 0.74\ x_{13} + 0.68\ x_{15} + 0.67\ x_{17} + 0.62\ x_{19} - 330{,}000 = 0$
$h_8 = 0.84\ x_{10} + 0.85\ x_{12} + 0.72\ x_{14} + 0.81\ x_{16} + 0.77\ x_{18} + 0.78\ x_{20} - 125{,}000 = 0$
$g_1 = x_3 + x_4 - 1{,}200{,}000 \leq 0$;
$g_2 = x_5 + x_6 - 1{,}000{,}000 \leq 0$;

$g_3 = x_9 + x_{10} + x_{15} + x_{16} - 190{,}000 \leq 0$

$g_4 = x_{11} + x_{12} + x_{17} + x_{18} - 240{,}000 \leq 0;$

$g_5 = x_{13} + x_{14} + x_{19} + x_{20} - 290{,}000 \leq 0;$

$-x_i \leq 0, \quad i = 1 \text{ to } 20$

2.22———————————————————————————————

Optimization of water canal. Design a water canal having a cross-sectional area of 150 m^2. Least construction costs occur when the volume of the excavated material equals the amount of material required for the dykes, i.e., $A_1 = A_2$ (see Figure E2.22). Formulate the problem to minimize the dug-out material A$_1$. Transcribe the problem into the standard design optimization model (created by V.K.Goel).
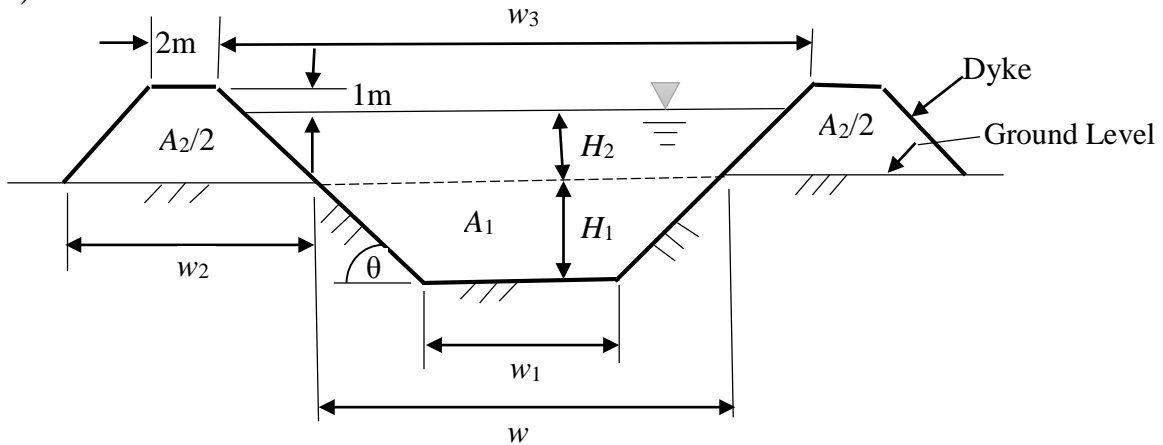

FIGURE E2.22 Cross section of a canal.

Solution

Given: The specific, required cross-sectional area of the canal, least construction costs occur when the volume of the excavated material is equivalent to the amount of material required for the two dykes, and the dimensions as shown in Figure E2.22.

Required: It is desired to minimize the dug-out material A$_1$.

Procedure: We follow the five step process to formulate the problem as an optimization problem.

Formulation 1:

Step 1: Problem Statement
　　Shown above

Step 2: Data and Information Collection
　　Shown above

Step 3: Definition of Design Variables
　　w, w_1, w_2, w_3, H_1 and H_2 (m) are chosen as design variables which are defined as shown in Figure E2.22.

Step 4: Optimization Criterion
　　Optimization criterion is to minimize the volume of excavation, and the cost function is defined as
　　$f = (w_1 + w)(H_1/2)$

Step 5: Formulation of Constraints

Cross-Sectional Area Constraint: $(w_1 + w_3)(H_1 + H_2 + 1)/2 = 150$ or
$h_1 = (w_1 + w_3)(H_1 + H_2 + 1)/2 - 150 = 0$;
Excavated Material Constraint: $(w_1 + w)(H_1/2) = (2)(w_2 + 2)(H_2 + 1)/2$ or
$h_2 = (w_1 + w)(H_1/2) - (w_2 + 2)(H_2 + 1) = 0$
The design variables are not independent; they are related as follows:

$$\tan\theta = \frac{H_1}{(w - w_1)/2} = \frac{H_2 + 1}{(w_2 - 2)/2} = \frac{H_1 + H_2 + 1}{(w_3 - w_1)/2}$$

So we get two more constraints from these relationships, as

$$h_3 = \frac{H_1}{w - w_1} - \frac{H_2 + 1}{w_2 - 2} = 0;$$

$$h_4 = \frac{H_1}{w - w_1} - \frac{H_1 + H_2 + 1}{w_3 - w_1} = 0$$

All the design variables must also be nonnegative:
$-w \le 0; \quad -w_1 \le 0; \quad -w_2 \le 0; \quad -w_3 \le 0; \quad -H_1 \le 0; \quad -H_2 \le 0$

Formulation 2:

Step 1: Problem Statement
Shown above

Step 2: Data and Information Collection
Shown above

Step 3: Definition of Design Variables
w_1, H_1, H_2 (m), and s (unitless) are chosen as design variables which are defined below in relation to Figure E2.22:

$$s = \tan\theta$$

$$w = w_1 + \frac{2H_1}{s}$$

$$w_3 = w_1 + \frac{2(H_1 + H_2 + 1)}{s}$$

$$w_2 = 2 + \frac{2(H_2 + 1)}{s}$$

Step 4: Optimization Criterion
Optimization criterion is to minimize the volume of excavation, and the cost function is defined as:

$$f = \frac{H_1\,(w + w_1)}{2} = \frac{H_1\,(\frac{2H_1}{s} + 2w_1)}{2}$$

Step 5: Formulation of Constraints

Cross-Sectional Area Constraint:

$$\frac{(w_1 + w_3)}{2}(H_1 + H_2 + 1) = 150$$

$$h_1 = \frac{[2w_1 + \frac{2\,(H_1 + H_2 + 1)}{s}]}{2} * (H_1 + H_2 + 1) - 150 = 0$$

$$A_1 = A_2$$

$$h_2 = \frac{H_1(2w_1 + \frac{2H_1}{s})}{2} - \frac{2\,(H_2 + 1)(2 + 2 + 2\,(H_2 + 1))}{2} = 0$$

All design variables must be non-negative :

$$w_1, H_1, H_2, s \geq 0$$

Formulation 3:

Step 1: Problem Statement

Shown above

Step 2: Data and Information Collection

Shown above

Step 3: Definition of Design Variables

A_1, A_2, w, w_1, w_2, w_3, H_1, H_2 (m), and s (unitless) are chosen as design variables which are defined above in Figure E2.22 and below:

$$s = \tan\theta$$

Step 4: Optimization Criterion

Optimization criterion is to minimize the volume of excavation, and the cost function is defined as:

$$f = A_1$$

Step 5: Formulation of Constraints

Cross-Sectional Area Constraint:

$$\frac{(w_1 + w_3)}{2}(H_1 + H_2 + 1) = 150$$

$$h_1 = \frac{(w_1 + w_3)}{2}(H_1 + H_2 + 1) - 150 = 0$$

$$A_1 = A_2$$

$$A_1 = \frac{(w + w_1)H_2}{2}$$

$$A_2 = \frac{2(2 + w_2)(H_2 + 1)}{2}$$

$$h_2 = \frac{(w + w_1)H_2}{2} - \frac{2(2 + w_2)(H_2 + 1)}{2} = 0$$

$$s = \frac{2H_1}{(w - w_1)} = \frac{2(H_2 + 1)}{(w_2 - 2)} = \frac{2(H_1 + H_2 + 1)}{(w_3 - w_1)}$$

$$h_3 = \frac{2H_1}{(w - w_1)} - \frac{2(H_2 + 1)}{(w_2 - 2)} = 0$$

$$h_4 = \frac{2H_1}{(w - w_1)} - \frac{2(H_1 + H_2 + 1)}{(w_3 - w_1)} = 0$$

$$h_5 = \frac{2(H_2 + 1)}{(w_2 - 2)} - \frac{2(H_1 + H_2 + 1)}{(w_3 - w_1)} = 0$$

All design variables ≥ 0

2.23

A cantilever beam is subjected to the point load P (kN), as shown in Fig. E2.23. The maximum bending moment in the beam is PL (kN·m) and the maximum shear is P (kN). Formulate the minimum mass design problem using a hollow circular cross section. The material should not fail under bending stress or shear stress. The maximum bending stress is calculated as

$$\sigma = \frac{PL}{I} R_0$$

where I = moment of inertia of the cross section. The maximum shearing stress is calculated as

$$\tau = \frac{P}{3I}(R_o^2 + R_0 R_i + R_i^2)$$

Transcribe the problem into the standard design optimization model (also use $R_0 \leq 40.0$ cm, $R_i \leq 40.0$ cm). Use the following data: $P = 14$ kN ; $L = 10$ m; mass density, $\rho = 7850$ kg/m³, allowable bending stress, $\sigma_a = 165$ MPa, Allowable shear stress, $\tau_a = 50$ MPa.
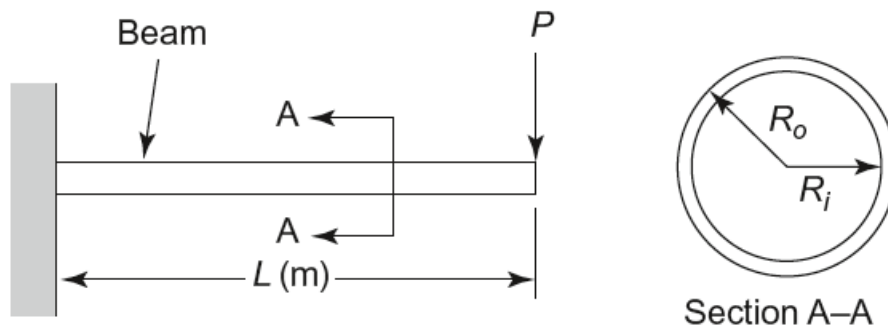

FIGURE E2.23 Cantilever beam.

Solution

Given: The equations to calculate maximum bending and shearing stress in the beam, the force applied to the beam, the length of the beam, the density of the beam, the maximum values of R_o and R_i, and the allowable bending and shear stress for the beam.

Required: It is desired to create a beam design, as shown in Figure E2.23, which will minimize the mass of the beam. The beam should not fail due to bending or shear at any point.

Procedure: We follow the five step process to formulate the problem as an optimization problem.

Step 1: Problem Statement
    Shown above

Step 2: Data and Information Collection
    Using kg, N and cm as units
    Given Data: (this data will change if different units are used)
    $P = 14$ kN $= 1.4 \times 10^4$ N
    $L = 10$ m $= 1000$ cm
    $\sigma_b = 165$ MPa $= 1.65 \times 10^4$ N/cm²;

$\tau_a = 50$ MPa $= 5000$ N/cm

$\rho = 7850$ kg/m$^3 = 7.85 \times 10^{-3}$ kg/cm^3;

Cross-sectional area of hollow tubes: $A = \pi(R_o^2 - R_i^2)$

Moment of inertia of a hollow tube is $I = \pi\left(R_o^4 - R_i^4\right)\big/4$

Maximum bending stress:

$$\sigma = \frac{PL}{I}R_0$$

Maximum shearing stress:

$$\tau = \frac{P}{3I}(R_o^2 + R_0 R_i + R_i^2)$$

In addition, it must be ensured that $R_o > R_i$ which can be imposed as a constraint on the wall thickness as $t \geq t_{min}$ with t_{min} as, say 0.5 cm.

Thickness: $t = R_o - R_i$

Step 3: Definition of Design Variables
R_o = outer radius of hollow tube, cm
R_i = inner radius of hollow tube, cm

FORMULATION 1: Using Intermediate Variables

Step 4: Optimization Criterion
Optimization criterion is to minimize mass of hollow tube, and the cost function is defined as
$f = \rho \pi A L$

Step 5: Formulation of Constraints
g_1 : bending stress should be smaller than the allowable bending stress; $\sigma \leq \sigma_a$

$g_1 = \sigma - \sigma_a \leq 0$

g_2 : shear stress smaller than allowable shear stress: $\tau \leq \tau_a$

$g_2 = \tau - \tau_a \leq 0$

$g_3 = R_o - 40 \leq 0$
$g_4 = R_i - 40 \leq 0$
$g_5 = -R_o \leq 0$
$g_6 = -R_i \leq 0$

$g_7 = t_{min} - t \leq 0$

FORMULATION 2: Using only Design Variables

Step 4: Optimization Criterion
Optimization criterion is to minimize mass of hollow tube, and the cost function is defined as

$$f = \rho \pi \left(R_o^2 - R_i^2 \right) L \ or$$

$$f = \rho \pi L \left(R_o^2 - R_i^2 \right) = \left(7.85 \times 10^3 \right) \pi (1000) \left(R_o^2 - R_i^2 \right) = 24.66 \left(R_o^2 - R_i^2 \right), \text{kg}$$

Step 5: Formulation of Constraints
g_1 : bending stress should be smaller than the allowable bending stress
g_2 : shear stress smaller than allowable shear stress

Using the standard form, we get

$$g_1 : 4Pl R_o \Big/ \pi \left(R_o^4 - R_i^4 \right) \le \sigma_b; \ \text{or} \ 4 \left(1.4 \times 10^4 \right) \left(10^3 \right) R_o \Big/ \pi \left(R_o^4 - R_i^4 \right) - 1.65 \times 10^4 \le 0; \text{or}$$

$$g_1 = 1.7825 \times 10^7 \, R_o \Big/ \left(R_o^4 - R_i^4 \right) - 1.65 \times 10^4 \le 0$$

$$g_2 : 4P \left(R_o^2 + R_o R_i + R_i^2 \right) \Big/ 3\pi \left(R_o^4 - R_i^4 \right) \le \tau_a; \ \text{or}$$

$$4 \left(1.4 \times 10^4 \right) \left(R_o^2 + R_o R_i + R_i^2 \right) \Big/ 3\pi \left(R_o^4 - R_i^4 \right) - 5000 \le 0; \ \text{or}$$

$$g_2 = 5941.78 \left(R_o^2 + R_o R_i + R_i^2 \right) \Big/ \left(R_o^4 - R_i^4 \right) - 5000 \le 0$$

$$g_3 = R_o - 40 \le 0;$$

$$g_4 = R_i - 40 \le 0;$$

$$g_5 = -R_o \le 0;$$

$$g_6 = -R_i \le 0$$

$$g_7 = t_{min} - (R_o - R_i) \le 0$$

2.24———————————————————————————————————————

Design a hollow circular beam-column, shown in Figure E2.24, for two conditions: When the axial tensile load P=50 (kN), the axial stress σ must not exceed an allowable value σ_a, and when P=0, deflection δ due to self-weight should satisfy the limit $\delta \le 0.001L$. The limits for dimensions are: thickness t=0.10 to 1.0 cm, mean radius R=2.0 to 20.0 cm, and $R/t \le 20$ (AISC, 2005). Formulate the minimum-weight design problem and transcribe it into the standard form. Use the following data: deflection δ=5wL^4/384EI; w=self-weight force/length (N/m); σ_a=250 MPa; modulus of elasticity E=210 GPa; mass density of beam material ρ=7800 kg/m^3; axial stress under load P, σ=P/A; gravitational constant g=9.80 m/s^2; cross-sectional area $A = 2\pi Rt$ (m^2); moment of inertia of beam cross-section I=$\pi R^3 t$ (m^4). Use Newton (N) and millimeters (mm) as units in the formulation.
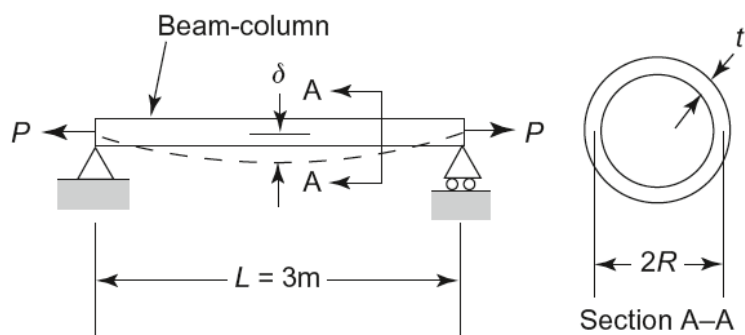

FIGURE E2.24 Graphic of a hollow circular beam-column.

Solution

Given: The maximum and minimum dimensions of t and R and the maximum ratio for R/t, the equations to calculate displacement, δ, axial stress, σ, cross-sectional area, moment of inertia, and the distributed force, w, the allowable axial stress, the modulus of elasticity, the mass density of the beam material, the gravitational constant, and the data shown in Figure E2.24. In addition, when P=50 kN, the stress must not exceed σ_a and when P=0 kN, deflection due to self-weight must satisfy $\delta \le 0.001L$.

Required: It is desired to create a beam design, as shown in Figure E2.24, which will minimize the mass of the beam, under the two conditions described above.

Procedure: We follow the five step process to formulate the problem as an optimization problem.

Step 1: Problem Statement
    Shown above

Step 2: Data and Information Collection
    Assuming that the wall is thin ($R \gg t$), the cross-sectional area and moment of inertia are:
    $$A = 2\pi Rt \; ; I = \pi R^3 t$$
    Use millimeter and Newton as the unit for length and force respectively, and the following data
    σ_a= 250 MPa = 250 N/mm^2; E = 210 GPa = 2.1$\times 10^5$ N/mm^2; g = 9.8 m/sec^2 (Note that g must have units of m/s^2 for correct evaluation of self-weight); ρ = 7800 kg/m^3 = 7.8$\times 10^{-6}$ kg/mm^3;
    L = 3 m = 3000 mm; P = 50 kN = 5$\times 10^4$ N;

Step 3: Definition of Design Variables

R = mean radius of the section, mm

t = wall thickness, mm

Step 4: Optimization Criterion

Optimization criterion is to minimize total weight of the beam-column, and the cost function is defined as

$f = mass\ (in\ kg) \times g(m/s^2) = (\rho L A)g = (2\rho L \pi R t)g$, N or

$f = 2(7.8 \times 10^{-6})(3000)\pi R t\ (9.8) = 1.44086 R t$, N

which is in the standard form.

Step 5: Formulation of Constraints

Axial stress (P/A) should not exceed the allowable stress (σ_a), i.e., $P/A \le \sigma_a$:

$g_1 = P/A - \sigma_a = P/(2\pi R t) - \sigma_a \le 0$

The deflection δ due to self-weight should be less than $0.001L$, i.e.,

$g_2 = \delta - 0.001L = 5wL^4/384EI - 0.001L \le 0$;

where w = self-weight per unit length $= \rho g A = 2\rho g \pi R t$, N/mm

$$g_2 = \frac{5(2\rho g \pi R t)L^4}{384E(\pi R^3 t)} - 0.001L \le 0, \quad or \quad \frac{10\rho g L^4}{384 E R^2} \le 0.001L$$

Other constraints are:

$g_3 = R/t - 20 \le 0$; $g_4 = -R + 20 \le 0$; $g_5 = R - 200 \le 0$; $g_6 = -t + 1 \le 0$; $g_7 = t - 10 \le 0$

or

$g_1: P/2\pi R t \le \sigma_a$; or $5.0 \times 10^4/2\pi R t - 250 \le 0$, or $g_1 = 7957.75/Rt - 250 \le 0$

$$g_2: \frac{10\rho g L^4}{384 E R^2} \le 0.001L, \quad or \quad \frac{10(7.8 \times 10^{-6})(9.8)(3000)^4}{384(2.1 \times 10^5)R^2} \le 0.001(3000)$$

Summarizing the constraints and rewriting in standard form, we get

$g_1 = 7957.7/Rt - 250 \le 0$

$g_2 = 767.8/R^2 - 3 \le 0$

$g_3 = R/t - 20 \le 0$

$g_4 = 20 - R \le 0$

$g_5 = R - 200 \le 0$

$g_6 = 1 - t \le 0$;

$g_7 = t - 10 \le 0$