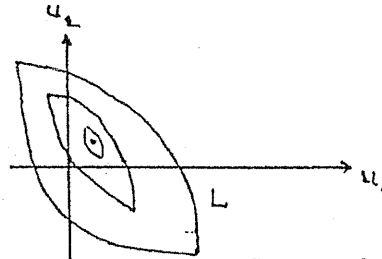# Chapter 1

1.1-1

a).

$$u^\star = -Q^{-1}S = \begin{bmatrix} -2 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

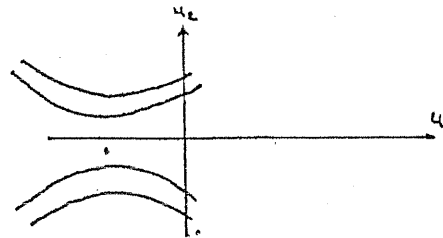$$L_{uu} = Q = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}.$$

Since Q is negative definite, $L_{uu} < 0$. Therefore, $u^\star$ is local maximum.

$$L_u = \frac{1}{2} \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -\frac{1}{2} u_1^2 + u_1 u_2 - u_2^2 + u_2$$

Optimum value of L, $L^\star = -1/2$. The gradient

$$L_u = Qu + S = \begin{bmatrix} u_2 - u_1 \\ u_1 - 2u_2 + 1 \end{bmatrix}.$$

b).

$$u^\star = -Q^{-1}S = -\frac{1}{3} \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$$

$$L_{uu} = Q = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}.$$

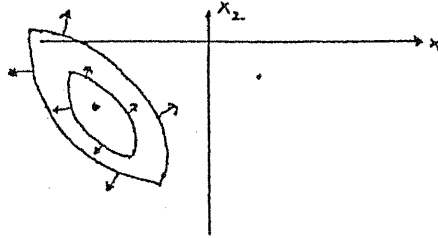Since $|Q| \not< 0$, $u^\star$ is a saddle point.

1.1-2

$$L(x_1, x_2) = x_1^2 - x_1 x_2 + x_2^2 + 3x_1$$

2

$$\left.\begin{array}{l}\frac{\partial L}{\partial x_1} = 2x_1 - x_2 + 3 = 0 \\ \frac{\partial L}{\partial x_2} = -x_1 + 2x_2 = 0\end{array}\right\} \Rightarrow x_1^* = -2, \quad x_2^* = -1.$$

$$L = -\frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

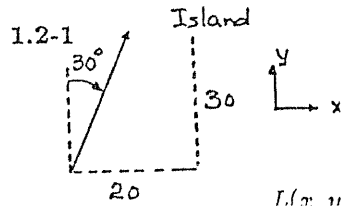$$L_{uu} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} > 0.$$

1.1-3

a). $u = \begin{bmatrix} x \\ y \end{bmatrix}, \quad f_u = \begin{bmatrix} 2x \\ 4y^3 \end{bmatrix}, \quad f_u|_{u=0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$

b).

$$f_{uu} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 12y^2 \end{bmatrix}, \quad f_{uu}|_{u=0} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

$|f_{uu}| = 0 \Rightarrow$ origin is a singular point.

c). $f(x, y) = x^2 + y^4$. $f(0, 0) = 0$, $x^2 > 0$ for all $x \neq 0$. $y^4 > 0$ for all $y \neq 0$.
$f(x, y) = x^2 + y^4 > 0$ for all $x \neq 0$ and $y \neq 0 \Rightarrow (0, 0)$ is a minimum point.

1.2-1

$$L(x, y) = \frac{1}{2}(x - 20)^2 + \frac{1}{2}(y - 30)^2$$

$$f(x, y) = \sqrt{3}x - y = 0,$$

$$H(x, y) = L + \lambda^T f = \frac{1}{2}(x - 20)^2 + \frac{1}{2}(y - 30)^2 + \lambda(\sqrt{3}x - y),$$

$$\left.\begin{array}{l}H_\lambda = \sqrt{3}x - y = 0 \\ H_x = x - 20 + \lambda\sqrt{3} = 0 \\ H_y = y - 30 - \lambda = 0\end{array}\right\} \Rightarrow \begin{array}{l}\lambda = 1.16 \\ x = 17.99 \\ y = 31.16\end{array}$$

Closest point: $(x, y) = (17.99, 31.16), distance = \sqrt{2L^*} = 2.32.$
Time $= \frac{\sqrt{x^2+y^2}}{10mph} \sim 36$ hours.

1.2-2

$$L = \min_{x,y}(d_1^2),$$
$$f = d_1^2 - d_2^2.$$

$$H = H(x,y,\lambda) = (x-x_1)^2 + (y-y_1)^2$$
$$+ \lambda \left[(x-x_1)^2 + (y-y_1)^2 - (x-x_2)^2 - (y-y_2)^2 \right]$$

$$\frac{\partial H}{\partial x} = 2(x-x_1) + 2\lambda(x-x_1) - 2\lambda(x-x_2) = 0$$

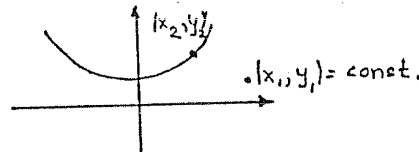$$\frac{\partial H}{\partial y} = 2(y-y_1) + 2\lambda(y-y_1) - 2\lambda(y-y_2) = 0$$

$$\frac{\partial H}{\partial \lambda} = (x-x_1)^2 + (y-y_1)^2 - (x-x_2)^2 - (y-y_2)^2 = 0$$

$$x = (1+\lambda)x_1 - \lambda x_2$$
$$y = (1+\lambda)y_1 - \lambda y_2$$
$$\lambda = -y_2, \quad x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}$$

1.2-3

$$H = \frac{1}{2}(x_1-x_2)^2 + \frac{1}{2}(y_1-y_2)^2 + \lambda \left(\frac{x_2^2}{a^2} - \frac{y_2^2}{b^2} - 1 \right)$$

$$H_{x_2} = -(x_1-x_2) + \frac{2\lambda x_2}{a^2} = 0,$$

$$H_{y_2} = -(y_1-y_2) + \frac{2\lambda y_2}{b^2} = 0,$$

$$H_\lambda = \frac{x_2^2}{a^2} - \frac{y_2^2}{b^2} - 1 = 0.$$

$$\lambda = \frac{a^2(x_1-x_2)}{2x_2} = -\frac{b^2(y_1-y_2)}{2y_2}.$$

From the above equation we obtain

$$\frac{x_2^2}{a^2} - \frac{y_2^2}{b^2} = 1,$$
$$a^2 x_1 y_2 + b^2 y_1 x_2 = (a^2 + b^2) x_2 y_2.$$

4

The above equations have two unknowns which can then be solved for.

1.2-4
a).

$$f(x,y) = x^2 + 3x - y - 6 = 0$$

$$L(x,y) = \frac{1}{2}(x-2)^2 + \frac{1}{2}(y-2)^2$$

$$H(x,y) = L + \lambda^T f = \frac{1}{2}(x-2)^2 + \frac{1}{2}(y-2)^2 + \lambda(x^2 + 3x - y - 6)$$

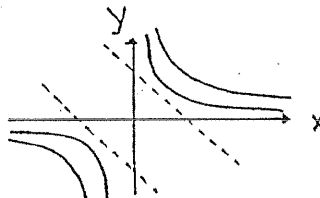$$H_\lambda = x^2 + 3x - y - 6 = 0$$

$$H_x = x - 2 + \lambda(2x+3) = 0$$

$$H_y = y - 2 - \lambda$$

$$y = \frac{3x+8}{2x+3}, \quad \lambda = \frac{2-x}{2x+3}, \quad 2x^3 + 9x^2 - 6x - 26 = 0,$$

b). By Newton's method ($x_0 = 1.5$, four iterations). $x \sim 1.7086$, $y \sim 2.0454$, $\lambda \sim 0.0454$, distance$=\sqrt{2L^\star} \sim 0.295$.

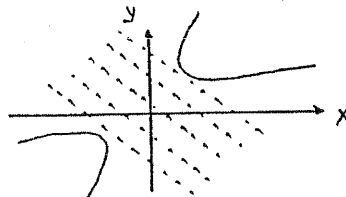**1.2-5**
a).

$$H(x,y) = xy + \lambda(2x + 2y - p)$$

$$H_\lambda = 2x + 2y - p = 0$$

$$H_x = y + 2\lambda = 0$$

$$H_y = x + 2\lambda = 0$$

$$\lambda = -p/8, \quad x = y = p/4.$$

b).

$$H(x,y) = 2x + 2y + \lambda(xy - a^2)$$

$$H_\lambda = xy - a^2 = 0$$

$$H_x = 2 + \lambda y = 0$$

$$H_y = 2 + \lambda x = 0$$

$$\lambda = \frac{\sqrt{2}}{a}, \quad x = y = \sqrt{2}a.$$

5

1.2-6

$$L = \frac{1}{2}x^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \frac{1}{2}u^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} u, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad R = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},$$

$$x = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} u,$$

$$f = x + \begin{bmatrix} -2 & -2 \\ -1 & 0 \end{bmatrix} u + \begin{bmatrix} -1 \\ -3 \end{bmatrix} = 0, \quad B = \begin{bmatrix} -2 & -2 \\ -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -1 \\ -3 \end{bmatrix},$$

$$H = \frac{1}{2}x^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \frac{1}{2}u^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} u + \lambda^T \left\{ x + \begin{bmatrix} -2 & -2 \\ -1 & 0 \end{bmatrix} u + \begin{bmatrix} -1 \\ -3 \end{bmatrix} \right\},$$

$$H_\lambda = x + \begin{bmatrix} -2 & -2 \\ -1 & 0 \end{bmatrix} u + \begin{bmatrix} -1 \\ -3 \end{bmatrix} = 0,$$

$$H_x = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \lambda = 0,$$

$$H_y = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} u + \begin{bmatrix} -2 & -2 \\ -1 & 0 \end{bmatrix} \lambda = 0,$$

$$u^* = -(R + B^T Q B)^{-1} B^T Q C = \begin{bmatrix} -3 \\ 13/5 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 1 \\ -0.4 & 1 \end{bmatrix},$$

$$x^* = -(I - BK)C = \begin{bmatrix} 1.8 \\ 6 \end{bmatrix}, \quad \lambda^* = -Qx = \begin{bmatrix} -1.8 \\ -12 \end{bmatrix},$$

$$L^* = \frac{1}{2}C^T \lambda = \begin{bmatrix} 0.9 \\ 1.8 \end{bmatrix}.$$

1.2-7

a).

$$\begin{aligned} K &= \left(R + B^T Q B\right)^{-1} B^T Q \\ U^* &= -KC \\ x^* &= -(I - BK)C \\ \lambda^* &= Q(I - BK)C \\ L^* &= \frac{1}{2}C^T Q[I - BK]C \end{aligned}$$

b).

$$\begin{aligned} S_0 &= Q - QB(B^T Q B + R^{-1})B^T Q = Q(I - BK) \\ K &= \left(R + B^T Q B\right)^{-1} B^T Q \\ 0 &= B^T Q - (R + B^T Q B)\,K \end{aligned}$$

6

Furthermore,

$$
\begin{aligned}
B^T Q(I - BK) - RK &= 0 \\
-K^T B^T Q(I - BK) + K^T RK &= 0 \\
Q(I - BK) - K^T B^T Q(I - BK) + K^T RK &= Q(I - BK)
\end{aligned}
$$

where $K^T B^T Q(I - BK) + K^T RK = 0$. So,

$$
\begin{aligned}
S_0 &= Q(I - BK) = (I - K^T B^T) Q(I - BK) + K^T RK \\
&= (I - BK)^T Q(I - BK) + K^T RK \\
L^\star &= \frac{1}{2} C^T S_0 C = \frac{1}{2} C^T (I - BK)^T \sqrt{Q}^T \sqrt{(Q)} (I - BK) C \\
&\quad + \frac{1}{2} C^T K^T \sqrt{R}^T \sqrt{R} K C \\
&= \frac{1}{2} \left[\sqrt{Q} (I - BK) C \right]^T \left[\sqrt{Q} (I - BK) C \right] + \frac{1}{2} \left(\sqrt{R} K C \right)^T \sqrt{R} K C
\end{aligned}
$$

c).

$$
\begin{aligned}
S_0 &= Q - QB \left(R + B^T QB \right)^{-1} B^T Q \\
&= \left(Q^{-1} + BR^{-1} B^T \right)^{-1}
\end{aligned}
$$

1.2-8
a).

$$
\begin{aligned}
L &= x^2 y^2 z^2, \quad f = x^2 + y^2 + z^2 - r^2 = 0 \\
H &= L + \lambda^T f = x^2 y^2 z^2 + \lambda^T (x^2 + y^2 + z^2 - r^2) \\
H_\lambda &= x^2 + y^2 + z^2 - r^2 = 0 \\
H_x &= 2x y^2 z^2 + 2x\lambda = 0 \tag{1} \\
H_y &= 2x^2 y z^2 + 2\lambda y = 0 \tag{2} \\
H_z &= 2x^2 y^2 z + 2z\lambda = 0 \tag{3}
\end{aligned}
$$

From (1)-(3), we have $x^2 = y^2 = z^2$. Now,

$$
\begin{aligned}
H_\lambda &= 3x^2 - r^2 = 0 \Rightarrow x^2 = \frac{r^2}{3} \\
L^\star &= \left(\frac{r^2}{3} \right)^3
\end{aligned}
$$

b).

$$
H = x^2 + y^2 + z^2 + \lambda \left(x^2 y^2 z^2 - \left(\frac{r^2}{3} \right)^3 \right)
$$

$$
\begin{aligned}
H_\lambda &= x^2 y^2 z^2 - \left(\frac{r^2}{3}\right)^3 = 0 \\
H_x &= 2x + 2x\lambda y^2 z^2 = 0 \Rightarrow y^2 z^2 = -1/\lambda \\
H_y &= 2y + 2y\lambda x^2 z^2 = 0 \Rightarrow x^2 z^2 = -1/\lambda \\
H_z &= 2z + 2z\lambda x^2 y^2 = 0 \Rightarrow x^2 y^2 = -1/\lambda.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
x^2 &= y^2 = z^2 \Rightarrow x^2 = r^2/3 \\
L^\star &= x^2 + y^2 + z^2 = r^2.
\end{aligned}
$$

c).

$$
\begin{aligned}
L &= b_1^2 b_2^2 b_3^2 \cdots b_n^2 \\
f &= b_1^2 + b_2^2 + b_3^2 + \cdots + b_n^2 - r^2 = 0 \\
H &= L + \lambda f \\
H_\lambda &= b_1^2 + b_2^2 + b_3^2 + \cdots + b_n^2 - r^2 = 0 \quad (4) \\
H_{b_1} &= 2b_1 b_2^2 b_3^2 \cdots b_n^2 + 2b_1 \lambda = 0 \quad (5) \\
&\vdots \qquad \vdots \\
H_{b_n} &= 2b_1^2 b_2^2 b_3^2 \cdots b_n + 2b_n \lambda = 0 \quad (6)
\end{aligned}
$$

From (4)-(6), we have

$$
\begin{aligned}
b_1^2 &= b_2^2 = b_3^2 = \cdots = b_n^2 \\
H_\lambda &= nb_1^2 - r^2 = 0 \Rightarrow b_1^2 = r^2/n \\
L^\star &= \left(\frac{r^2}{n}\right)^n = \left(\frac{b_1^2 + b_2^2 + \cdots + b_n^2}{n}\right)^n \\
b_1^2 b_2^2 \cdots b_n^2 &\leq \left(\frac{b_1^2 + b_2^2 + \cdots + b_n^2}{n}\right)^n
\end{aligned}
$$

Let $a_i = b_i^2$. Hence, $a_i > 0$ and $\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^n$ is the maximum value of $a_1 a_2 a_3 \cdots a_n$ i.e.,

$$
\begin{aligned}
a_1 a_2 a_3 \cdots a_n &\leq \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^n \\
(a_1 a_2 a_3 \cdots a_n)^{\frac{1}{n}} &\leq \frac{(a_1 + a_2 + \cdots + a_n)}{n}
\end{aligned}
$$

8

1.2-9

$$
\begin{aligned}
f_1(x,y,z) &= 3x + 2y + z - 1 = 0 \\
f_2(x,y,z) &= x + 2y - 3z - 4 = 0 \\
L(x,y,z) &= \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2
\end{aligned}
$$

Let

$$
f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad w = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.
$$

$$
H = \frac{1}{2}w^T w + \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 & -1 \\ 1 & 2 & -3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
$$

$$
\begin{aligned}
H_{\lambda_1} &= 3x + 2y + z - 1 = 0 \\
H_{\lambda_2} &= x + 2y - 3z + 4 = 0 \\
H_x &= y + 3\lambda_1 + \lambda_2 = 0 \quad &(7) \\
H_y &= y + 2\lambda_1 + 2\lambda_2 = 0 \quad &(8) \\
H_z &= z + \lambda_1 - 3\lambda_2 = 0 \quad &(9)
\end{aligned}
$$

From (7)-(9) we obtain

$$
\begin{aligned}
x &= -3\lambda_1 - \lambda_2 \\
y &= -2\lambda_1 - 2\lambda_2 \\
z &= -\lambda_1 + 3\lambda_2
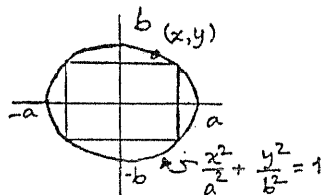\end{aligned}
$$

and

$$
\begin{aligned}
7\lambda_1 + 2\lambda_2 &= -\frac{1}{2} \\
2\lambda_1 + 7\lambda_2 &= -2
\end{aligned}
$$

result in $\lambda_1 = \frac{1}{90}$, $\lambda_2 = -\frac{13}{95}$. Hence, $x = \frac{23}{90}$, $y = \frac{5}{9}$, $z = -\frac{79}{90}$.

1.2-10

a). Here we have,

$$L(x,y) = 4(x+y)$$

$$f(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

$$H(x,y) = L + \lambda^T f = 4x + 4y + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$

$$H_x = 4 + \frac{2\lambda}{a^2}x = 0 \tag{11}$$

$$H_y = 4 + \frac{2\lambda}{b^2}y = 0 \tag{12}$$

$$H_\lambda = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \tag{13}$$

Equations (11)-(13) result in

$$\lambda = -2\sqrt{a^2 + b^2}$$

$$x = \frac{a^2}{\sqrt{a^2 + b^2}}$$

$$y = \frac{b^2}{\sqrt{a^2 + b^2}}$$

$$L^f_{yy} = H_{yy} - 2\frac{f_y}{f_x}H_{xy} + \frac{f_y^2}{f_x^2}H_{xx}$$

$$= 2\frac{\lambda}{b^2} + \frac{2\lambda y^2 a^2}{x^2 b^4} = \frac{2\lambda}{b^2}\left[1 + \frac{a^2 y^2}{b^2 x^2}\right] < 0, \quad iff \ \lambda < 0$$

b).

$$L(x,y) = 4xy, \quad f(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

$$H(x,y) = L + \lambda^T f = 4xy + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$

$$H_x = 4y + \frac{2\lambda}{a^2}x = 0 \tag{14}$$

$$H_y = 4 + \frac{2\lambda}{b^2}y = 0 \tag{15}$$

$$H_\lambda = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \tag{16}$$

Equations (14)-(16) result in

$$\lambda = -2ab,$$

10

$$x = \frac{a}{\sqrt{2}},$$

$$y = \frac{b}{\sqrt{2}}$$

which gives the optimum area

$$L^\star = \frac{ab}{2}.$$

This area is the maximum since

$$L_{yy}^f = H_{yy} - 2\frac{f_y}{f_x}H_{xy} + \frac{f_y^2}{f_x^2}H_{xx}$$

$$= -4\frac{a}{b}\left[1 + \frac{ay}{bx}\right]^2 < 0, \quad (a > 0, b > 0).$$