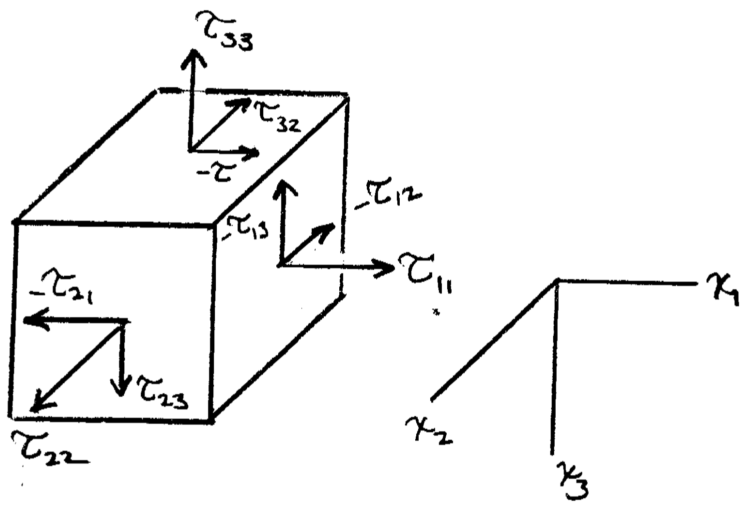
CHAPTER 1

1.1



1.2

2. (a) $T_i^{(\nu)} = \tau_{ij}\,\nu_j$

$T_1^{(\nu)} = \tau_{11}(0.11) + \tau_{12}(0.35) + \tau_{13}(0.93) = 180$ psi

$T_2^{(\nu)} = -2450$ psi

$T_3^{(\nu)} = -140$ psi

(b) To find component in direction of unit vector $\hat{\varepsilon}$, compute

$$\vec{T}^{(\hat{\varepsilon})} = \vec{T}^{(\nu)} \cdot \hat{\varepsilon} = T_i^{(\nu)} \cdot \varepsilon_i = 180(.33) - 2450(.90) - 140(.284)$$

$$= -2185 \text{ psi}$$

1.3 If there was a body-couple vector $\vec{M}$, with components M_i, it would follow that

$$M_i = \varepsilon_{ijk}\,\tau_{jk}\;;\quad \therefore\; M_1 = \varepsilon_{123}\,\tau_{23} + \varepsilon_{132}\,\tau_{32}$$

$$\therefore\; \tau_{32} = \tau_{23} - M_1 = 100 - 200 = -100 \text{ psi}$$

1.4 From Eq. (1.11), $\tau'_{pq} = a_{pj}\, a_{qi}\, \tau_{ji}$, corresponding to the transformation $x'_i = a_{ij}\, x_j$. For the rotation given, the transformation matrix is

$$a_{ij} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

It is a straightforward exercise to obtain the stress matrix τ'_{pq}:

$$\tau'_{pq} = \begin{pmatrix} 137 & -137 & 0 \\ -137 & 63 & 0 \\ 0 & 0 & 500 \end{pmatrix}$$

Note that in calculating the above, the only non-zero terms in the τ_{ij} matrix are τ_{11}, $\tau_{12} = \tau_{21}$, τ_{33}. Thus

$$\tau'_{pq} = (a_{p1}\, a_{q1})\,\tau_{11} + (a_{p2}\, a_{q1} + a_{p1}\, a_{q2})\,\tau_{12} + (a_{p3}\, a_{q3})\,\tau_{33}\,.$$

1.5

(a) The principal stresses are found from the equation

$$\tau^3 - I_\tau \tau^2 + II_\tau \tau - III_\tau = 0\,.$$

In the present example the invariants of the given stress state are easily computed as

$$I_\tau = \tau_{kk} = 0\,; \quad III_\tau = |\tau_{ij}| \equiv 0\,.$$

$$II_\tau = \begin{vmatrix} -1000 & 1000 \\ 1000 & 1000 \end{vmatrix} + |0| + |0| = -2\text{x}10^6\ (\text{psi})^2$$

$$\tau^3 - 2\text{x}10^6\,\tau = 0 \;\rightarrow\; \begin{cases} \tau = 0 \\ \tau^2 = 2 \times 10^6 \end{cases}$$

$$\therefore\ \tau_1 = \sqrt{2} \times 10^3 \text{ psi},\quad \tau_2 = 0\,,\quad \tau_3 = -\sqrt{2} \times 10^3 \text{ psi}$$

(b) The array of principal values can be written

$$\tau_i = \begin{pmatrix} \sqrt{2}\times 10^3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2}\times 10^3 \end{pmatrix}$$

From this array we can again compute the invariants, to compare with the results in (a). Trivially, $I_\tau = III_\tau = 0$. And

$$II_\tau = \begin{vmatrix} \sqrt{2}\times 10^3 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} \sqrt{2}\times 10^3 & 0 \\ 0 & \sqrt{2}\times 10^3 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & -\sqrt{2}\times 10^3 \end{vmatrix}$$

$= -2\times 10^6$. Thus, agreement!

(c) To obtain the direction cosines, need to solve $(\tau_{ij} - \sigma\delta_{ij})\nu_j = 0$. See note below.**

For $\tau_1 = \sqrt{2}\times 10^3$, after some algebra: $\overset{1}{\nu}_3 = 0$

$$(1+\sqrt{2})\overset{1}{\nu}_1 = \overset{1}{\nu}_2$$

But $\overset{1}{\nu}_i\,\overset{1}{\nu}_i = 1 \rightarrow \overset{1}{\nu}_1 = \dfrac{\sqrt{2}-1}{\sqrt{4-2\sqrt{2}}}$ $\quad \overset{1}{\nu}_2 = \dfrac{1}{\sqrt{4-2\sqrt{2}}}$

$$\overset{1}{\nu}_3 = 0$$

For $\tau_2 = 0$, get $\overset{2}{\nu}_1 = \overset{2}{\nu}_2 = 0$, $\overset{2}{\nu}_3 = 1$.

For $\tau_3 = -\sqrt{2}\times 10^3$, get $\overset{3}{\nu}_1 = \dfrac{-1-\sqrt{2}}{\sqrt{4+2\sqrt{2}}}$

$$\overset{3}{\nu}_2 = \frac{1}{\sqrt{4+2\sqrt{2}}}, \quad \overset{3}{\nu}_3 = 0.$$

** Since $\tau_{13} = \tau_{23} = \tau_{33} = 0$, these equations reduce simply to the set:

$$(\tau_{11} - \tau)\nu_1 + \tau_{12}\nu_2 = 0$$
$$\tau_{12}\nu_1 + (\tau_{22} - \tau)\nu_2 = 0$$
$$\tau\nu_3 = 0.$$

The orthogonality relations, for $\tau_1 \neq \tau_2 \neq \tau_3$ that must be satisfied are

$$0 = \overset{1}{\nu}_j \overset{2}{\nu}_j = 0 + 0 + 0 = 0$$

$$0 = \overset{1}{\nu}_j \overset{3}{\nu}_j = \frac{\sqrt{2}-1}{\sqrt{4-2\sqrt{2}}}\,\frac{-(1+\sqrt{2})}{\sqrt{4+2\sqrt{2}}} + \frac{1}{\sqrt{4-2\sqrt{2}}}\,\frac{1}{\sqrt{4+2\sqrt{2}}} = \frac{1-1}{\sqrt{4-2\sqrt{2}\;4+2\sqrt{2}}} = 0$$

$$0 = \overset{2}{\nu}_j \overset{3}{\nu}_j = 0 + 0 + 0 = 0$$

$\therefore$ Have obtained correct principal stresses and directions.

1.6

The stress quadric is $\xi_i \xi_j \tau_{ij} = \pm d^2$.

(a) Uniaxial tension: $\tau_{11} = c^2 =$ constant; other $\tau_{ij} = 0$.

$$\therefore \xi^2_1 \tau_{11} = \xi^2_1 c^2 = \pm d^2 = + d^2$$

$$\left.\xi_1 = \pm \frac{d}{c}\right\}$$ represents a pair of plane surfaces.

(b) Plane stress: Let $\tau_{13} = \tau_{23} = \tau_{33} = 0$.

$$\therefore \xi_i \xi_j \tau_{ij} = \xi^2_1 \tau_{11} + 2\xi_1 \xi_2 \tau_{12} + \xi^2_2 \tau_{22} = \pm d^2 .$$

$$\therefore \xi_1^2 + 2\frac{\tau_{12}}{\tau_{11}} \xi_1 \xi_2 + \frac{\tau_{22}}{\tau_{11}} \xi_2^2 = \pm \frac{d^2}{\tau_{11}}$$

This is clearly the equation of a closed curve in the ξ_1, ξ_2 plane, or a cylinder in ξ_1, ξ_2, ξ_3 space .

(c) For simple shear: only $\tau_{12} = \tau_{21} = c \neq 0$. Then

$$\xi_i \xi_j \tau_{ij} = 2c\,\xi_1 \xi_2 = \pm d^2 .$$

$\therefore \xi_1 \xi_2 = \pm \dfrac{d^2}{2c}$, which is the eq. of a rectangular hyperbola in $\xi_1 \xi_2$ space.

1.7 To find $(\tau_{nn})_{oct}$ we can use Cauchy's formula. We assume that the x_i are coincident with the axes of principal stress τ_i. As the plane under consideration is at equal orientation to the principal axes, the components ν_i of the plane's normal $\vec{\nu}$ must be equal. Therefore, $\nu_i = 1/\sqrt{3}$, $i = 1, 2, 3$. Then

$$(\tau_{nn})_{oct} = \vec{T}^{(\nu)} \cdot \vec{\nu} = T_i^{(\nu)} \nu_i = \tau_{ij} \nu_i \nu_j$$
$$= \tau_i \nu_i^2 = \frac{1}{3}(\tau_1 + \tau_2 + \tau_3) .$$

The total stress vector on the inclined plane can be resolved into two components, the normal component $T^{(n)} \equiv (\tau_{nn})_{oct}$, and the shear component $T^{(s)} \equiv \tau_{oct}$. If $\vec{T}^{(\nu)}$ is the total stress vector on the inclined plane

$$(T^{(s)})^2 = \vec{T}^{(\nu)} \cdot \vec{T}^{(\nu)} - (\tau_{nn})^2_{oct}.$$

But $\vec{T}^{(\nu)} = \tau_{ij} \nu_j$ which is Cauchy's law, derived from Newton's law, thus

$$\vec{T}^{(\nu)} \cdot \vec{T}^{(\nu)} = \tau_{ij} \nu_j \tau_{ik} \nu_k = \tau_1^2 \nu_1^2 + \tau_2^2 \nu_2^2 + \tau_3^2 \nu_3^2 .$$

Thus $\tau^2_{oct} = \frac{1}{3}(\tau_1^2 + \tau_2^2 + \tau_3^2) - \frac{1}{9}(\tau_1 + \tau_2 + \tau_3)^2$

$$\therefore \tau^2_{oct} = \tau_1^2(\tfrac{1}{3} - \tfrac{1}{9}) + \tau_2^2(\tfrac{1}{3} - \tfrac{1}{9}) + \tau_3^2(\tfrac{1}{3} - \tfrac{1}{9}) - \tfrac{2}{9}(\tau_1\tau_2 + \tau_1\tau_3 + \tau_2\tau_3).$$
$$= \frac{1}{9}[2\tau_1^2 + 2\tau_2^2 + 2\tau_3^2 - 2\tau_1\tau_2 - 2\tau_1\tau_3 - 2\tau_2\tau_3]$$

$$\therefore \tau^2_{oct} = \frac{1}{9}[(\tau_1 - \tau_2)^2 + (\tau_2 - \tau_3)^2 + (\tau_1 - \tau_3)^2]$$

For the uniaxial tension test, let $\tau_1 = Y$, $\tau_2 = \tau_3 = 0$.
Then $\tau^2_{oct} = \frac{2}{9} Y^2$. Then the criterion of Mises-Hencky follows as

$$(\tau_1 - \tau_2)^2 + (\tau_2 - \tau_3)^2 + (\tau_1 - \tau_3)^2 = 2 Y^2$$

or, more simply, $\tau_{oct} = \frac{\sqrt{2}}{3} Y$ for yielding.

1.8 For principal stresses, $I_\tau = \tau_1 + \tau_2 + \tau_3$, and

$II_\tau = \tau_1\tau_2 + \tau_1\tau_3 + \tau_2\tau_3$. Then

$$9\tau_{oct}^2 = 2\tau_1^2 + 2\tau_2^2 + 2\tau_3^2 - 2II_\tau$$
$$= 2(\tau_1 + \tau_2 + \tau_3)^2 - 4\tau_1\tau_2 - 4\tau_1\tau_3 - 4\tau_2\tau_3 - 2II_\tau$$

$$\therefore\ 9\tau_{oct}^2 = 2(I_\tau)^2 - 6II_\tau\ .$$

1.9

The Green tensor is defined as (Eq. (1.32a)):

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \xi_k}{\partial x_i}\frac{\partial \xi_k}{\partial x_j} - \delta_{ij}\right).$$

Then by straightforward substitution, the following strain tensor arrays are found:

(a) Simple dilatation: $\varepsilon_{ij} = \begin{pmatrix} \frac{1}{2}(\lambda^2 - 1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(b) Pure deformation: $\varepsilon_{ij} = \begin{pmatrix} \frac{1}{2}(\lambda_1^2 - 1) & 0 & 0 \\ 0 & \frac{1}{2}(\lambda_2^2 - 1) & 0 \\ 0 & 0 & \frac{1}{2}(\lambda_3^2 - 1) \end{pmatrix}$

(c) Cubical dilatation: $\varepsilon_{ij} = \frac{1}{2}(\lambda^2 - 1)\,\delta_{ij}$.

(d) Simple shear: $\varepsilon_{ij} = \begin{pmatrix} 0 & \frac{1}{2}\Gamma & 0 \\ \frac{1}{2}\Gamma & \frac{1}{2}\Gamma^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Note in (d) the normal strain ε_{22} that occurs in the nonlinear theory for simple shear. Expressed in terms of stresses, this is often called the "normal stress effect".

1.10

Here $u_i = \xi_i - x_i = u_i(x_j)$, and $2\varepsilon_{ij}^{\text{linear}} = (u_{i,j} + u_{j,i})$

(a) Simple dilatation: $\varepsilon_{ij}^{\text{linear}} = \begin{pmatrix} \lambda - 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(b) Pure deformation: $\varepsilon_{ij}^{\text{linear}} = \begin{pmatrix} \lambda_1 - 1 & 0 & 0 \\ 0 & \lambda_2 - 1 & 0 \\ 0 & 0 & \lambda_3 - 1 \end{pmatrix}$

(c) Cubical dilatation: $\varepsilon_{ij}^{\text{linear}} = (\lambda - 1)\delta_{ij}$

(d) Simple shear: $\varepsilon_{ij}^{\text{linear}} = \begin{pmatrix} 0 & \Gamma/2 & 0 \\ \Gamma/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

For physical interpretations, consider first simple dilatation. Then

$\dfrac{d\xi_1}{dx_1} = \lambda$, $\dfrac{d\xi_2}{dx_2} = \dfrac{d\xi_3}{dx_3} = 1$. Thus we may note that

$\lambda = \dfrac{\text{deforming length}}{\text{initial length}} \equiv$ <u>stretch ratio</u>

Also, equally obviously, from case (d), we can identify Γ as the <u>shear angle</u>.

1.11

The initial volume of an element V is $dx_1 dx_2 dx_3$. The deformed volume is $V^* = d\xi_1\, d\xi_2\, d\xi_3$. Hence

$V^* = (\lambda_1 dx_1)(\lambda_2 dx_2)(\lambda_3 dx_3) = \lambda_1 \lambda_2 \lambda_3 V$, using the information of Prob. 9, part (b), for pure deformation.

From the definition of the Green strain tensor,

$$(ds^*)^2/(ds)^2 = 1 + [2\varepsilon_{ij}\, dx_i\, dx_j/(ds)^2]$$

Let $ds = dx_1$ $\therefore$ $(ds^*)^2/(dx_1)^2 = \lambda_1^2$. Thus

$$\lambda_1^2 = 1 + 2\varepsilon_{11} \rightarrow \lambda_1 = \sqrt{1 + 2\varepsilon_{11}}$$

$$\therefore \quad \lambda_i = \sqrt{1 + 2\,\varepsilon_{ii}}\ , \quad i = 1,2,3; \ (\text{no sum!})$$

Then, from above,

$$V^*/V = \sqrt{1 + 2\,\varepsilon_{11}}\ \sqrt{1 + 2\,\varepsilon_{22}}\ \sqrt{1 + 2\,\varepsilon_{33}}$$

If the strains are infinitesimal, $2\,\varepsilon_{ij} = u_{i,j} + u_{j,i} << 1$.

Then

$$V^*/V \cong (1 + \varepsilon_{11})(1 + \varepsilon_{22})(1 + \varepsilon_{33}) \cong 1 + \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

$$\therefore \quad V^*/V - 1 = \varepsilon_{ii} = \nabla \cdot \vec{U}$$

[Note, can also get $\lambda_i^2 = 1 + 2\,\varepsilon_{ii}$ result from Problem (9) , part (b).]

1.12 Clearly $\varepsilon_{ij} = 0$ implies $u_{i,j} + u_{j,i} = 0$. Thus

$$\frac{\partial u}{\partial x} = 0 \rightarrow u = u_1(y,z); \quad \frac{\partial v}{\partial y} = 0 \rightarrow v = v_1(x,z); \quad \frac{\partial w}{\partial z} = 0 \rightarrow w = w_1(x,y).$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \rightarrow \frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x} = 0 \ \text{①}; \quad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0 \rightarrow \frac{\partial u_1}{\partial z} + \frac{\partial w_1}{\partial x} = 0 \ \text{②}$$

$$\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \rightarrow \frac{\partial v_1}{\partial z} + \frac{\partial w_1}{\partial y} = 0 \ \text{③} .$$

The implications of the vanishing normal strains can then be used with equations ① - ③ as follows:

$$\text{①} \rightarrow u_1 = -y\,\frac{\partial v_1}{\partial x} + f(z) \ ; \quad \text{②} \rightarrow u_1 = -\,z\,\frac{\partial w_1}{\partial x} + g(y)$$

$$\text{①} \rightarrow v_1 = -x\,\frac{\partial u_1}{\partial y} + h(z) \ ; \quad \text{③} \rightarrow v_1 = -\,z\,\frac{\partial w_1}{\partial y} + i(x)$$

$$\text{②} \rightarrow w_1 = -x\,\frac{\partial u_1}{\partial z} + j(y) \ ; \quad \text{③} \rightarrow w_1 = -\,y\,\frac{\partial v_1}{\partial z} + k(x)$$

Then $\dfrac{\partial u_1}{\partial x} = 0 \qquad \dfrac{\partial^2 v_1}{\partial x^2} = \dfrac{\partial^2 w_1}{\partial x^2} = 0$.

$$\therefore \quad v_1 = A(z)x + B(z) \ , \quad w_1 = C(y)x + D(y)$$

Also $\dfrac{\partial v_1}{\partial y} = 0 \rightarrow u_1 = E(z)y + F(z) \ , \quad w_1 = G(x)y + H(x)$.

And $\dfrac{\partial w_1}{\partial z} = 0 \rightarrow u_1 = I(y)z + J(y) \ , \quad v_1 = K(x)z + L(x)$.

$$\therefore \quad u_1(y,z) = E(z)y + F(z) = I(y)z + J(y)$$

$$v_1(x,z) = K(x)z + L(x) = A(z)x + B(z)$$

$$w_1(x,y) = G(x)y + H(x) = C(y)x + D(y) .$$

From the equations for $u_1(y,z)$, $E(z)y - I(y)z = J(y) - F(z)$

$\therefore$ $E(z) = E_1 = \text{const.}$, $I(y) = I_1 = \text{const.}$

$\therefore$ $u_1(y,z) = E_1 y + I_1 z$; similarly,

$v_1(x,z) = K_1 z + A_1 x$;

$w_1(x,y) = C_1 x + G_1 y$.

But $\dfrac{\partial u_1}{\partial y} + \dfrac{\partial v_1}{\partial x} = E_1 + A_1 = 0 \longrightarrow E_1 = -A_1$.

Similarly, $C_1 = -I_1$, $G_1 = -K_1$.

$$\therefore \quad u(x,y,z) = 0 + E_1 y + I_1 z$$

$$v(x,y,z) = -E_1 x + 0 + K_1 z$$

$$w(x,y,z) = -I_1 x - K_1 y + 0 .$$

If we write $\vec{w} = w_1\hat{i} + w_2\hat{j} + w_3\hat{k} = -K_1\hat{i} + I_1\hat{j} - E_1\hat{k}$, and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, we can easily see that

$$\vec{u} = \vec{\omega} \times \vec{r} .$$

A constant translational displacement can always be added. If it is denoted as $\vec{u}_o$, with $\partial \vec{u}_o / \partial x_i = 0$ for all i , the most general rigid body displacement can then be written as

$$\vec{u} = \vec{u}_o + \vec{\omega} \times \vec{r} .$$

1.13 We have just shown (Problem 11), that the volumetric change is calculated as

$$V^*/V - 1 = \varepsilon_{ii} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} .$$

Thus in terms of the deviator, $(V^*/V - 1)_{\text{Deviator}} = \bar{\varepsilon}_{ii}$, or

$(\Delta V/V)_{\text{Deviator}} = \varepsilon_{ii} - \frac{1}{3} e \, \delta_{ii} = \varepsilon_{ii} - \frac{3}{3} e \equiv 0$. Hence the dilatation of the deviator is identically zero.

To determine the invariants, consider the array of principal strains, since $e\delta_{ij}$ affects only the diagonal terms. Thus the principal deviator strains are (with $e = \frac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$)

$$\bar{\varepsilon}_{ij} = \begin{pmatrix} (\varepsilon_1 - \frac{1}{3}e) & 0 & 0 \\ 0 & (\varepsilon_2 - \frac{1}{3}e) & 0 \\ 0 & 0 & (\varepsilon_3 - \frac{1}{3}e) \end{pmatrix} = \begin{pmatrix} \bar{\varepsilon}_1 & 0 & 0 \\ 0 & \bar{\varepsilon}_2 & 0 \\ 0 & 0 & \bar{\varepsilon}_3 \end{pmatrix}$$

Then the invariants are determined from $\left| \bar{\varepsilon}_{ij} - \lambda\delta_{ij} \right| = 0$.

Then $\lambda^3 - I_{\bar{\varepsilon}}\lambda^2 + II_{\bar{\varepsilon}}\lambda - III_{\bar{\varepsilon}} = 0$

$$\therefore \quad I_{\bar{\varepsilon}} = \bar{\varepsilon}_1 + \bar{\varepsilon}_2 + \bar{\varepsilon}_3 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - 3e \equiv 0 .$$

$$II_{\bar{\varepsilon}} = \bar{\varepsilon}_1\bar{\varepsilon}_3 + \bar{\varepsilon}_2\bar{\varepsilon}_3 + \bar{\varepsilon}_1\bar{\varepsilon}_2 = (\varepsilon_1 - e)(\varepsilon_3 - e)$$

$$+ (\varepsilon_2 - e)(\varepsilon_3 - e) + (\varepsilon_1 - e)(\varepsilon_2 - e)$$

$$= -\frac{1}{6}[(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_1 - \varepsilon_3)^2 + (\varepsilon_2 - \varepsilon_3)^2]$$

$$III_{\bar{\varepsilon}} = \bar{\varepsilon}_1\bar{\varepsilon}_2\bar{\varepsilon}_3 = (\varepsilon_1 - \frac{e}{3})(\varepsilon_2 - \frac{e}{3})(\varepsilon_3 - \frac{e}{3})$$

1.14 The transformation law for a second-order tensor is

$$A'_{ij} = a_{ip}a_{jq}A_{pq} .$$

Then $\lambda\delta'_{ij} = a_{ip}a_{jq}\lambda\delta_{pq}$

or $\lambda\delta'_{ij} = \lambda a_{ip}a_{jp} \equiv \lambda\delta_{ij}$

Thus the equation of a second-order, isotropic tensor has been identified.

1.15 For rotation about the x_3 axis,

$$a_{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ; \quad \left\{ \begin{array}{l} \text{Note, isotropy transformations:} \\ A'_{ij} = a_{ip}a_{jq}A_{pq} \equiv A_{ij} \end{array} \right\}$$

$$\therefore \quad A_{23} = a_{22}a_{33}A_{23} = -A_{23} \longrightarrow A_{23} = 0$$

$$A_{32} = a_{33}a_{22}A_{32} = -A_{32} \longrightarrow A_{32} = 0$$

$$A_{13} = a_{11}\, a_{33}\, A_{13} = -A_{13} \longrightarrow A_{13} = 0$$

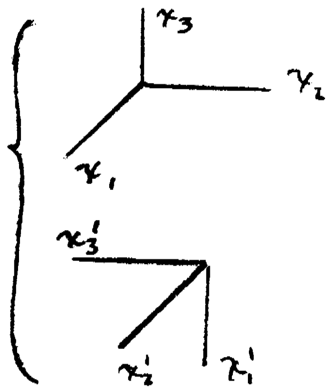
$$A_{31} = a_{33}\, a_{11}\, A_{31} = -A_{31} \longrightarrow A_{31} = 0\,.$$

Similarly for the rotation about the x_1 axis,

$$a_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} ; \qquad \begin{matrix} A_{12} = -A_{12} = 0 \\ A_{21} = -A_{21} = 0 \end{matrix}$$

Now a rotation about the x_1 axis of $90°$, followed by a rotation of $90°$ about the x_2 axis, produces

$$a_{ij} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$



Then

$$\left.\begin{aligned} A_{11} &= a_{13}\, a_{13}\, A_{33} = A_{33} \\ A_{22} &= a_{21}\, a_{21}\, A_{11} = A_{11} \\ A_{33} &= a_{32}\, a_{32}\, A_{22} = A_{22} \end{aligned}\right\} \longrightarrow A_{11} = A_{22} = A_{33}$$

Thus $\qquad A_{11} = A_{22} = A_{33} = \lambda\,.$

$\therefore \quad A_{ij} = \lambda\,\delta_{ij}$ is the most general second order isotropic tensor.

1.16

For $D_{ijkl} = \lambda \delta_{ij}\, \delta_{kl} + \beta \delta_{ik}\, \delta_{jl} + \gamma \delta_{il}\, \delta_{jk}$, carry out the fourth order transformation-

$$\begin{aligned} D'_{ijkl} &= \lambda a_{im}\, a_{jn}\, a_{ko}\, a_{lp}\, \delta_{mn}\, \delta_{op} + \beta a_{im}\, a_{ko}\, a_{jn}\, a_{lp}\, \delta_{mo}\, \delta_{np} \\ &\qquad + \gamma\, a_{im}\, a_{lp}\, a_{jn}\, a_{ko}\, \delta_{mp}\, \delta_{no} \\ &= \lambda a_{im}\, a_{jm}\, a_{kp}\, a_{lp} + \beta\, a_{im}\, a_{km}\, a_{jn}\, a_{ln} + \gamma\, a_{im}\, a_{lm}\, a_{jn}\, a_{kn} \\ &= \lambda \delta_{ij}\, \delta_{kl} + \beta\, \delta_{ik}\, \delta_{jl} + \gamma \delta_{il}\, \delta_{jk} \equiv D_{ijkl}\,. \quad \text{QED.} \end{aligned}$$