Chapter 1

Overview of Finite Element Method

1.1

$S^{(\ell)} = n\, r \quad$ with $\quad r = 2\, R \sin\left(\frac{1}{2} \cdot \frac{2\pi}{n}\right) = 2 R \sin \frac{\pi}{n}$

$S^{(u)} = n\, s \quad$ with $\quad s = 2\, R \tan\left(\frac{1}{2} \cdot \frac{2\pi}{n}\right) = 2 R \tan \frac{\pi}{n}$

Using series expansions of $\sin\theta$ and $\tan\theta$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - + \cdots$$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2}{15}\theta^5 + \cdots$$

we find

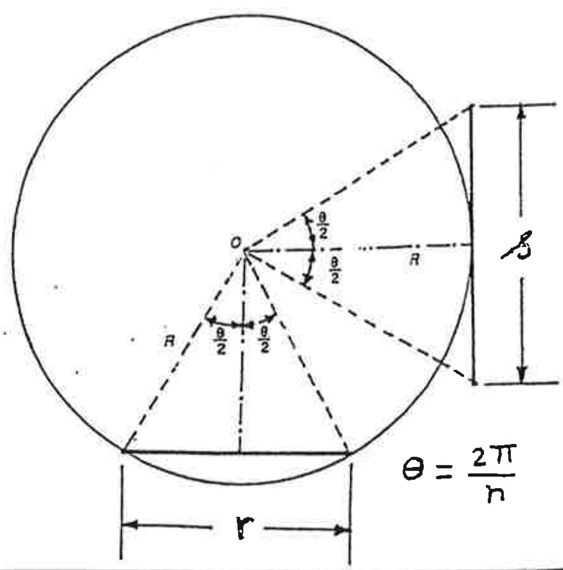
$$S^{(\ell)} = 2nR\left(\frac{\pi}{n} - \frac{1}{6}\frac{\pi^3}{n^3} + - \cdots\right)$$

$$= 2\pi R - \frac{\pi^3 R}{3 n^2} + - \cdots$$

and

$$S^{(u)} = 2nR\left(\frac{\pi}{n} + \frac{1}{3}\frac{\pi^3}{n^3} + \cdots\right)$$

$$= 2\pi R + \frac{2}{3}\frac{\pi^3 R}{n^2} + \cdots$$



Since the true value of S is given by $2\pi R$, we have

$$S^{(\ell)} \le S \le S^{(u)}$$

1.2

For $R = 1$:

n	$S^{(l)} = 2nR\sin\frac{\pi}{n}$	$S^{(u)} = 2nR\tan\frac{\pi}{n}$
3	5.1962	10.3923
4	5.6569	8.0000
5	5.8779	7.2654
6	6.0000	6.9282
7	6.0744	6.7420
8	6.1229	6.6274
9	6.1564	6.5515
10	6.1803	6.4984
11	6.1981	6.4598
12	6.2117	6.4308

1.3 From Fig. 1.4 (shown below), the area of the circle corresponding to the inscribed polygon is given by

$$A^{(l)} = n\left(\frac{1}{2} AB \times OP\right) = \frac{n}{2}\left(2R\sin\frac{\pi}{n}\right)\left(R\cos\frac{\pi}{n}\right)$$

$$= \frac{nR^2}{2}\left(2\sin\frac{\pi}{n}\cos\frac{\pi}{n}\right) = \frac{1}{2} nR^2 \sin\frac{2\pi}{n}$$

$$= \frac{1}{2} nR^2 \frac{2\pi}{n}\left(\frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}\right) = \pi R^2 \left(\frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}\right)$$

As $n \to \infty$, $\frac{2\pi}{n} \to 0$ and $A^{(l)} \Rightarrow \pi R^2 = A_{correct}$

Similarly, the area of the circle corresponding to the circumscribed polygon can be expressed as

$$A^{(u)} = n\left(\frac{1}{2} CD \times OQ\right) = \frac{n}{2}\left(2R\tan\frac{\pi}{n}\right)R$$

$$= nR^2\tan\frac{\pi}{n} = nR^2\frac{\pi}{n}\left(\frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}}\right) = \pi R^2\left(\frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}}\right)$$

As $n \to \infty$, $\frac{\pi}{n} \to 0$ and $A^{(u)} \Rightarrow \pi R^2 = A_{correct}$

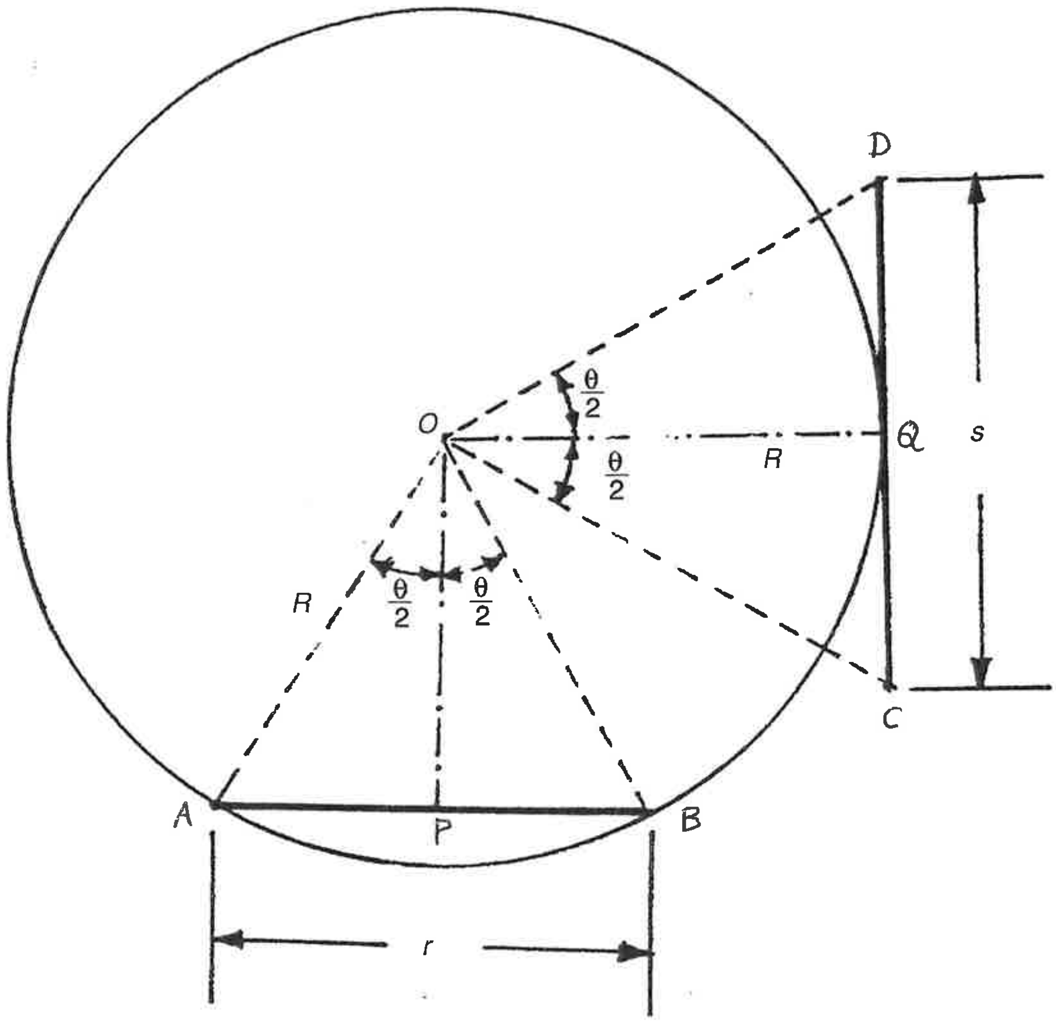
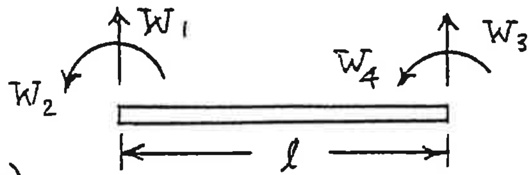


Figure 1.4.

1.4



$$w(x) = W_1^{(e)} \cdot \frac{1}{\ell^3}\left(2x^3 - 3\ell x^2 + \ell^3\right) + W_2^{(e)} \cdot \frac{1}{\ell^2}\left(x^3 - 2\ell x^2 + \ell^2 x\right) + W_3^{(e)} \cdot \frac{1}{\ell^3}\left(3\ell x^2 - 2x^3\right) + W_4^{(e)} \cdot \frac{1}{\ell^2}\left(x^3 - \ell x^2\right)$$

$$[K^{(e)}] = \frac{2EI}{\ell^3}\begin{bmatrix} 6 & 3\ell & -6 & 3\ell \\ 3\ell & 2\ell^2 & -3\ell & \ell^2 \\ -6 & -3\ell & 6 & -3\ell \\ 3\ell & \ell^2 & -3\ell & 2\ell^2 \end{bmatrix} = [\underset{\sim}{K}]$$

$$\vec{W}^{(e)} = \begin{Bmatrix} W_1^{(e)} \\ W_2^{(e)} \\ W_3^{(e)} \\ W_4^{(e)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ W_3 \\ W_4 \end{Bmatrix} = \underset{\sim}{\vec{W}}$$

$$\vec{P}^{(e)} = \begin{Bmatrix} P_1^{(e)} \\ P_2^{(e)} \\ P_3^{(e)} \\ P_4^{(e)} \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P \\ 0 \end{Bmatrix} = \underset{\sim}{\vec{P}}$$

$[\underset{\sim}{K}]\,\underset{\sim}{\vec{W}} = \underset{\sim}{\vec{P}}$ gives

$$\frac{2EI}{\ell^3}\begin{bmatrix} 6 & 3\ell & -6 & 3\ell \\ 3\ell & 2\ell^2 & -3\ell & \ell^2 \\ -6 & -3\ell & 6 & -3\ell \\ 3\ell & \ell^2 & -3\ell & 2\ell^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P \\ 0 \end{Bmatrix} \qquad (1)$$

The third and fourth equations in (1) are given by

$$\frac{2EI}{\ell^3}\left(6\,W_3 - 3\ell\,W_4\right) = P \qquad (2)$$

$$\frac{2EI}{\ell^3}\left(-3\ell\,W_3 + 2\ell^2\,W_4\right) = 0 \qquad (3)$$

Solution of Eqs. (2) and (3) is:

$$W_3 = \frac{P\ell^3}{3EI}\ , \qquad W_4 = \frac{P\ell^2}{2EI}$$

The deflection within the element can be expressed as

$$w(x) = \frac{P}{3EI}\left(3\ell x^2 - 2x^3\right) + \frac{P}{2EI}\left(x^3 - \ell x^2\right)$$

$$\frac{d^2 w}{dx^2}(x) = \frac{P}{3EI}\left(6\ell - 12x\right) + \frac{P}{2EI}\left(6x - 2\ell\right) = \frac{P}{EI}\left(\ell - x\right)$$

Bending moment $= M(x) = EI\,\dfrac{d^2w}{dx^2} = P(\ell - x)$

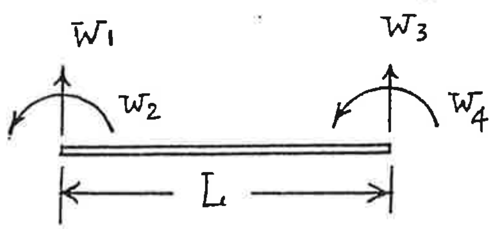
stress in the element $= \sigma(x) = \dfrac{M(x)\cdot y}{I}$

where y = distance from neutral axis.

$$\therefore\ \sigma(x) = \frac{P}{I}(\ell - x)\cdot y \qquad ; \qquad \ell \equiv L$$

1.5

$$[K^{(e)}] = [\underset{\sim}{K}] = \frac{2EI}{L^3}\begin{array}{c} \\ \begin{bmatrix} 6 & 3L & -6 & 3L \\ 3L & 2L^2 & -3L & L^2 \\ -6 & -3L & 6 & -3L \\ 3L & L^2 & -3L & 2L^2 \end{bmatrix} \\ \begin{array}{cccc} W_1 & W_2 & W_3 & W_4 \end{array}\end{array}\begin{array}{c} W_1 \\ W_2 \\ W_3 \\ W_4 \\ \ \end{array}$$



$$[K]\vec{W} = \vec{P} \quad \text{where} \quad [K] = \frac{2EI}{L^3}\begin{bmatrix} 6 & -3L \\ -3L & 2L^2 \end{bmatrix}$$

with $\vec{W} = \begin{Bmatrix} W_3 \\ W_4 \end{Bmatrix}$ and $\vec{P} = \begin{Bmatrix} 0 \\ M_0 \end{Bmatrix}$

These equations can be rewritten as

$$6\,W_3 - 3L\,W_4 = 0\,, \qquad -3L\,W_3 + 2L^2\,W_4 = \frac{M_o\,L^3}{2EI}$$

Solution: $\quad W_3 = \dfrac{M_o L^2}{2EI}\,, \quad W_4 = \dfrac{M_o L}{EI}$

Bending moment in element $= M(x) = EI\,\dfrac{d^2 w}{dx^2}$

with $\quad w(x) = W_3 \cdot \dfrac{1}{L^3}\left(3Lx^2 - 2x^3\right) + W_4 \cdot \dfrac{1}{L^2}\left(x^3 - Lx^2\right)$

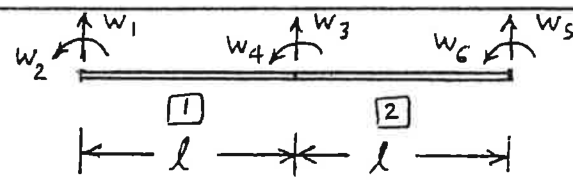
and $\quad \dfrac{d^2 w}{dx^2}(x) = W_3 \cdot \dfrac{1}{L^3}\left(6L - 12x\right) + \dfrac{W_4}{L^2}\left(6x - 2L\right)$

$$= \frac{M_o L^2}{2EI}\cdot\frac{1}{L^3}\left(6L - 12x\right) + \frac{M_o L}{EI}\cdot\frac{1}{L^2}\left(6x - 2L\right)$$

$$= \frac{M_o}{EI}$$

$$\sigma(x) = \text{stress} = \frac{M(x)\cdot y}{I} = \left(EI\,\frac{d^2 w}{dx^2}\right)\frac{y}{I} = \frac{M_o \cdot y}{I}$$

1.6 Assembling the element stiffness matrices by retaining only the non-zero degrees of freedom, we obtain



$W_1 = W_2 = W_5 = 0$

$$[K] = \frac{2EI}{l^3}\begin{array}{c} \begin{array}{ccc} W_3 & W_4 & W_6 \end{array} \\ \begin{bmatrix} 6+6 & -3l+3l & 3l \\ -3l+3l & 2l^2+2l^2 & l^2 \\ 3l & l^2 & 2l^2 \end{bmatrix} \end{array} \begin{array}{c} W_3 \\ W_4 \\ W_6 \end{array} \; ; \; l = \frac{L}{2}$$

$$= \frac{16EI}{L^3}\begin{bmatrix} 12 & 0 & \frac{3}{2}L \\ 0 & L^2 & \frac{1}{4}L^2 \\ \frac{3}{2}L & \frac{1}{4}L^2 & \frac{1}{2}L^2 \end{bmatrix}$$

Equilibrium equations:

$$\frac{4EI}{L^3}\begin{bmatrix} 48 & 0 & 6L \\ 0 & 4L^2 & L^2 \\ 6L & L^2 & 2L^2 \end{bmatrix}\begin{Bmatrix} W_3 \\ W_4 \\ W_6 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \\ 0 \end{Bmatrix} \qquad (1)$$

Scalar form of Eq. (1):

$$48\, W_3 + 6L\, W_6 = \frac{P L^3}{4EI} \qquad (2)$$

$$4L^2 W_4 + L^2 W_6 = 0 \quad \Rightarrow \quad W_4 = -\frac{W_6}{4} \qquad (3)$$

$$6L\, W_3 + L^2 W_4 + 2L^2 W_6 = 0 \qquad (4)$$

Eqs. (3) and (4) give

$$W_3 = -\frac{L^2}{6L} W_4 - \frac{2L^2}{6L} W_6 = -\frac{7}{24} L\, W_6 \qquad (5)$$

Eqs. (2), (5) and (3) yield

$$W_3 = \frac{7}{768}\frac{PL^3}{EI}, \quad W_4 = \frac{1}{128}\frac{PL^2}{EI}, \quad W_6 = -\frac{PL^2}{32\,EI} \qquad (6)$$

Bending moment in element $= M(x) = EI \dfrac{d^2 w}{dx^2}$

stress in element $= \sigma(x) = \dfrac{M(x)\cdot y}{I} = E\, y \dfrac{d^2 w}{dx^2} \qquad (7)$

$$\sigma_{xx}^{(1)} = -E\,y \left[\frac{1}{L^3}(6L - 12x)\frac{7PL^3}{768\,EI} + \frac{1}{L^2}(6x - 2L)\frac{PL^2}{128\,EI} \right]$$

$$= -y\,\frac{P}{I}\left(\frac{5L - 8x}{128}\right) \qquad (7)$$

$$\sigma_{xx}^{(2)} = -y\,E \left[\frac{1}{L^3}(12x - 6L)\frac{7PL^3}{768\,EI} + \frac{1}{L^2}(6x - 4L)\frac{PL^2}{128\,EI} + \frac{1}{L^2}(6x - 2L)\left(-\frac{PL^2}{32\,EI}\right) \right]$$

$$= y\,\frac{P}{I}\left(\frac{x + 2L}{32}\right) \qquad (8)$$

1.7 Assembled stiffness matrix, [K], is same as the one derived in Problem 1.6. Equilibrium equations:

$$\frac{4EI}{L^3}\begin{bmatrix} 48 & 0 & 6L \\ 0 & 4L^2 & L^2 \\ 6L & L^2 & 2L^2 \end{bmatrix}\begin{Bmatrix} W_3 \\ W_4 \\ W_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ M_0 \\ 0 \end{Bmatrix} \qquad (1)$$

Solution of Eqs. (1):

$$W_3 = \frac{M_0 L^2}{128\ EI}, \quad W_4 = \frac{5\ M_0 L}{64\ EI}, \quad W_6 = -\frac{M_0 L}{16\ EI}$$

stress element 1:

$$\sigma(x) = -E y \frac{d^2 w}{dx^2} = -E y \left[\frac{W_3}{l^3}(6l - 12x) + \frac{W_4}{l^2}(6x - 2l) \right]$$

$$= -E y \left[\frac{M_0 L^2}{128\ EI} \cdot \frac{8}{L^3}(3L - 12x) + \frac{5 M_0 L}{64 EI} \cdot \frac{4}{L^2}(6x - L) \right]$$

$$= -\frac{M_0\, y}{8 I L}(9x - L)$$

Stress in element 2:

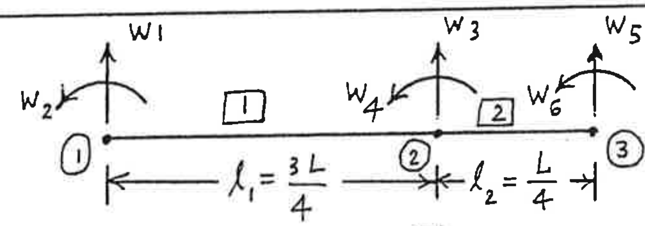
$$\sigma(x) = -E y \left[\frac{W_3}{\ell^3}(12x - 6\ell) + \frac{W_4}{\ell^2}(6x - 4\ell) + \frac{W_6}{\ell^2}(6x - 2\ell)\right]$$

$$= -E y \left[\frac{M_0 L^2}{128 EI}\cdot\frac{8}{L^3}(12x - 3L) + \frac{5 M_0 L}{64 EI}\cdot\frac{4}{L^2}(6x - 2L) - \frac{M_0 L}{16 EI}\cdot\frac{4}{L^2}\cdot(6x - L)\right]$$

$$= -\frac{9 M_0\, y}{16 I L}(2x - L)$$

1.8

2 element idealization ⇒



$$[K^{(e)}] = \frac{2EI}{\ell_e^3}\begin{bmatrix} 6 & 3\ell_e & -6 & 3\ell_e \\ 3\ell_e & 2\ell_e^2 & -3\ell_e & \ell_e^2 \\ -6 & -3\ell_e & 6 & -3\ell_e \\ 3\ell_e & \ell_e^2 & -3\ell_e & 2\ell_e^2 \end{bmatrix}$$

where ℓ_e = length of element e.

Boundary conditions: $W_1 = W_2 = W_5 = W_6 = 0$.

Considering only the free degrees of freedom, the assembled stiffness matrix of the system can be obtained as

$$[K] = 2EI \begin{array}{c} \begin{array}{cc} W_3 & W_4 \end{array} \\ \left[\begin{array}{c|c} \frac{6}{\ell_1^3} + \frac{6}{\ell_2^3} & -\frac{3}{\ell_1^2} + \frac{3}{\ell_2^2} \\ \hline -\frac{3}{\ell_1^2} + \frac{3}{\ell_2^2} & \frac{2}{\ell_1} + \frac{2}{\ell_2} \end{array}\right] \end{array} \begin{array}{c} W_3 \\ W_4 \end{array} = \begin{bmatrix} \left(\frac{7168}{9}\frac{EI}{L^3}\right) & \left(\frac{256}{3}\frac{EI}{L^2}\right) \\ \left(\frac{256}{3}\frac{EI}{L^2}\right) & \left(\frac{64}{3}\frac{EI}{L}\right) \end{bmatrix}$$

Equilibrium equations: $[K]\vec{W} = \vec{P}$

i.e.

$$\begin{bmatrix} \frac{7168}{9}\frac{EI}{L^3} & \frac{256}{3}\frac{EI}{L^2} \\ \frac{256}{3}\frac{EI}{L^2} & \frac{64}{3}\frac{EI}{L} \end{bmatrix} \begin{Bmatrix} W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$

Solution is:

$$W_3 = \frac{9}{4096}\frac{PL^3}{EI}, \quad W_4 = -\frac{9}{1024}\frac{PL^2}{EI}$$

Stress in element 1:

$$w^{(1)}(x) = \frac{9}{4096}\frac{PL^3}{EI}\cdot\frac{1}{l_1^3}\left(3l_1 x^2 - 2x^3\right) - \frac{9}{1024}\frac{PL^2}{EI} * \frac{1}{l_1^2}\left(x^3 - l_1 x^2\right)$$

$$= \frac{P}{192\,EI}\left(\frac{9}{2}Lx^2 - 5x^3\right)$$

$$\sigma^{(1)}(x) = \frac{M^{(1)}y}{I} = Ey\frac{d^2w^{(1)}}{dx^2} = Ey\cdot\frac{P}{192EI}\left(9L - 30x\right)$$

$$= \frac{P}{64\,I}\left(3L - 10x\right)y.$$

Stress in element 2:

$$w^{(2)}(x) = \frac{9}{4096}\frac{PL^3}{EI}\left(\frac{1}{l_2^3}\right)\left(2x^3 - 3l_2x^2 + l_2^3\right) - \frac{9}{1024}\frac{PL^2}{EI}\left(\frac{1}{l_2^2}\right)\left(x^3 - 2l_2x^2 + l_2^2x\right)$$

$$= \frac{9}{64}\frac{P}{EI}\left(x^3 - \frac{L}{4}x^2 - \frac{L^2}{16}x + \frac{L^3}{64}\right)$$

$$\sigma^{(2)}(x) = \frac{M^{(2)}y}{I} = Ey\frac{d^2w^{(2)}}{dx^2} = Ey\cdot\frac{9P}{128EI}\left(12x - L\right)$$

$$= \frac{9P}{128\,I}\left(12x - L\right)y.$$

1.9 $A(x) = A_0 e^{-x/l}$

$$\pi = \text{strain energy} = \frac{1}{2}\int_0^l \sigma \in A(x)\cdot dx$$

$$= \frac{1}{2}\int_0^l E\left(\frac{\partial u}{\partial x}\right)^2 A(x)\, dx \quad \text{with} \quad u(x) = \left(1-\frac{x}{l}\right)U_1 + \left(\frac{x}{l}\right)U_2$$

$$\pi = \frac{1}{2}\int_0^l E\left(\frac{U_2-U_1}{l}\right)^2 A_0 e^{-x/l}\, dx$$

$$= \frac{1}{2} E A_0 \left(\frac{U_2-U_1}{l}\right)^2 \int_0^l e^{-x/l}\, dx$$

$$= \frac{1}{2} E A_0 \left(\frac{U_2-U_1}{l}\right)^2 (0.6321\, l) = \frac{1}{2}\{U_1 \quad U_2\}[K]\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}$$

with

$$[K] = \frac{0.6321\, E A_0}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$T = \text{kinetic energy} = \frac{1}{2}\int_0^l \rho\left(\frac{\partial u}{\partial t}\right)^2 A(x)\cdot dx$$

$$= \frac{1}{2}\int_0^l \{\dot{U}_1 \quad \dot{U}_2\}\begin{bmatrix} \left(1-\frac{x}{l}\right)^2 & \left(1-\frac{x}{l}\right)\frac{x}{l} \\ \left(1-\frac{x}{l}\right)\frac{x}{l} & \left(\frac{x}{l}\right)^2 \end{bmatrix}\begin{Bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{Bmatrix} A_0 e^{-x/l}\cdot dx$$

$$= \frac{1}{2}\dot{\vec{U}}^T [M]\, \dot{\vec{U}}$$

Using $\int_0^l e^{-x/l}\cdot dx = 0.6321\, l$,

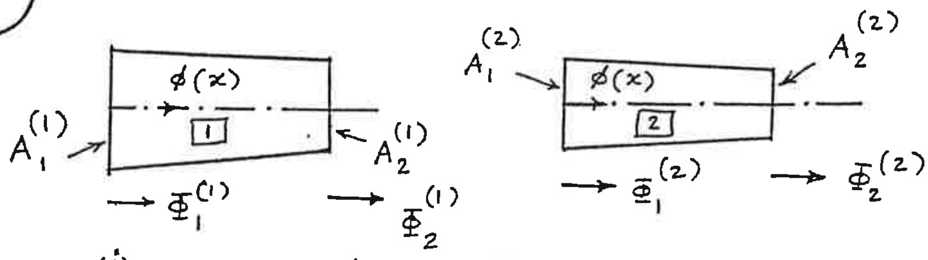
$\int_0^l \frac{x}{l} e^{-x/l}\cdot dx = 0.2642\, l$,

and $\int_0^l \frac{x^2}{l^2} e^{-x/l}\cdot dx = 0.1605\, l$, we obtain

$$[M] = \rho A_0 l \begin{bmatrix} 0.2642 & 0.1037 \\ 0.1037 & 0.1605 \end{bmatrix}$$

where ρ = density and l = length of element.

1.10 Step 1. Idealization:



$\ell^{(i)} = 5$ cm, $E^{(i)} = 2 \times 10^7$ N/cm^2 ; $i = 1, 2$

$A_1^{(1)} = 2$ cm^2, $A_2^{(1)} = A_1^{(2)} = 1.5$ cm^2, $A_2^{(2)} = 1$ cm^2

Step 2. Interpolation model:

$$\phi(x) = \left[\left(1 - \frac{x}{\ell^{(e)}}\right) \quad \left(\frac{x}{\ell^{(e)}}\right) \right] \begin{Bmatrix} \Phi_1^{(e)} \\ \Phi_2^{(e)} \end{Bmatrix}$$

Step 3. Element equations:

strain energy of element $e = \int_0^{\ell^{(e)}} \left(\frac{1}{2} \sigma^{(e)} \epsilon^{(e)}\right) A^{(e)} \, dx$

with $\sigma^{(e)} = E^{(e)} \epsilon^{(e)}$, $\epsilon^{(e)} = \frac{d\phi}{dx} = \frac{\Phi_2^{(e)} - \Phi_1^{(e)}}{\ell^{(e)}}$

Here $A^{(e)}(x) = A_1^{(e)} + \frac{A_2^{(e)} - A_1^{(e)}}{\ell^{(e)}} x$

Hence

$$\pi^{(e)} = \text{strain energy} = \frac{E^{(e)}}{2} \left(\frac{\Phi_2^{(e)} - \Phi_1^{(e)}}{\ell^{(e)}} \right)^2 \int_0^{\ell^{(e)}} \left\{ A_1^{(e)} + \frac{A_2^{(e)} - A_1^{(e)}}{\ell^{(e)}} x \right\} dx$$

$$= \frac{A_{av}^{(e)} E^{(e)}}{2 \ell^{(e)}} \left(\Phi_1^{(e)^2} + \Phi_2^{(e)^2} - 2 \Phi_1^{(e)} \Phi_2^{(e)} \right)$$

where $A_{av}^{(e)}$ = average area $= \frac{A_1^{(e)} + A_2^{(e)}}{2}$

Expressing $\pi^{(e)}$ as $\pi^{(e)} = \frac{1}{2} \vec{\Phi}^{(e)^T} [K^{(e)}] \vec{\Phi}^{(e)}$,

we obtain

$$[K^{(e)}] = \frac{A_{av}^{(e)} E^{(e)}}{\ell^{(e)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{with} \quad \vec{\Phi}^{(e)} = \begin{Bmatrix} \Phi_1^{(e)} \\ \Phi_2^{(e)} \end{Bmatrix}$$

$$A_{av}^{(1)} = \frac{2+1.5}{2} = 1.75 \text{ cm}^2, \quad A_{av}^{(2)} = \frac{1.5+1}{2} = 1.25 \text{ cm}^2$$

$$[K^{(1)}] = 10^6 \begin{bmatrix} 7 & -7 \\ -7 & 7 \end{bmatrix}, \quad [K^{(2)}] = 10^6 \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix}$$

Step 4. Assembly of equations:

$$[\underset{\sim}{K}] = 10^6 \begin{bmatrix} 7 & -7 & 0 \\ -7 & 7+5 & -5 \\ 0 & -5 & 5 \end{bmatrix}$$

$$\vec{\underset{\sim}{\Phi}} = \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{Bmatrix}, \quad \vec{\underset{\sim}{P}} = \begin{Bmatrix} P_1 \\ 0 \\ 1 \end{Bmatrix}, \quad P_1 = \text{reaction at node 1}$$

$$[\underset{\sim}{K}] \vec{\underset{\sim}{\Phi}} = \vec{\underset{\sim}{P}} \qquad (1)$$

Step 5. Apply boundary conditions and solve equations:

Since $\Phi_1 = 0$, we delete first row and first column in Eqs. (1) to obtain

$$[K] \vec{\Phi} = \vec{P} \Rightarrow 10^6 \begin{bmatrix} 12 & -5 \\ -5 & 5 \end{bmatrix} \begin{Bmatrix} \Phi_2 \\ \Phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \qquad (2)$$

Solution of Eqs. (2) gives

$$\Phi_2 = 1.428 \times 10^{-7} \text{ cm}, \quad \Phi_3 = 3.428 \times 10^{-7} \text{ cm}$$

Step 6. Find stresses:

$$\sigma^{(1)} = E^{(1)} \epsilon^{(1)} = E^{(1)} \frac{\Phi_2^{(1)} - \Phi_1^{(1)}}{l^{(1)}} = (2 \times 10^7)\left(\frac{1.428 \times 10^{-7} - 0}{5}\right)$$

$$= 0.5712 \text{ N/cm}^2$$

$$\sigma^{(2)} = E^{(2)} \epsilon^{(2)} = E^{(2)} \frac{\Phi_2^{(2)} - \Phi_1^{(2)}}{l^{(2)}}$$

$$= (2 \times 10^7)\left(\frac{3.428 \times 10^{-7} - 1.428 \times 10^{-7}}{5}\right) = 0.8 \text{ N/cm}^2$$

1.11 (a) Element characteristic matrices:

$$c_1 = \frac{\pi d_1^4}{128\,\mu_1\,\ell_1} = \frac{\pi\,(5^4)}{128\,(1.6\times10^{-6})(1000)} = 9.5874\times10^{3}\ \text{in}^5/\text{lb-sec}$$

$$c_2 = \frac{\pi d_2^4}{128\,\mu_2\,\ell_2} = \frac{\pi\,(2^4)}{128\,(1.6\times10^{-6})(1500)} = 0.1636\times10^{3}\ \text{in}^5/\text{lb-sec}$$

$$c_3 = \frac{\pi d_3^4}{128\,\mu_3\,\ell_3} = \frac{\pi\,(4^4)}{128\,(1.6\times10^{-6})(2000)} = 1.9635\times10^{3}\ \text{in}^5/\text{lb-sec}$$

Pipe element 1 with nodes 1 and 2:

$$[k^{(1)}] = c_1\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 9587.4\ \begin{array}{c} \begin{array}{cc}1 & 2\end{array}\\ \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\end{array}\begin{array}{c}\\ 1\\ 2\end{array}$$

Pipe element 2 with nodes 2 and 3:

$$[k^{(2)}] = c_2\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 1636.0\ \begin{array}{c} \begin{array}{cc}2 & 3\end{array}\\ \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\end{array}\begin{array}{c}\\ 2\\ 3\end{array}$$

Pipe element 3 with nodes 2 and 4:

$$[k^{(3)}] = c_3\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 1963.5\ \begin{array}{c} \begin{array}{cc}2 & 4\end{array}\\ \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\end{array}\begin{array}{c}\\ 2\\ 4\end{array}$$

Assembled characteristic matrix:

$$[\underset{\sim}{K}] = \begin{bmatrix} 9587.4 & -9587.4 & 0 & 0\\ -9587.4 & \begin{pmatrix}9587.4\\ +163.6\\ +1963.5\end{pmatrix} & -163.6 & -1963.5\\ 0 & -163.6 & 163.6 & 0\\ 0 & -1963.5 & 0 & 1963.5 \end{bmatrix}$$

System equations:

$$\begin{bmatrix} 9587.4 & -9587.4 & 0 & 0 \\ -9587.4 & 11714.5 & -163.6 & -1963.5 \\ 0 & -163.6 & 163.6 & 0 \\ 0 & -1963.5 & 0 & 1963.5 \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} \quad (1)$$

where Q_i is the external flow rate at node i.

Since the pressures at nodes 1, 3 and 4 are known, p_2 is the unknown pressure in Eq. (1) with $Q_2 = 0$.

By using the known values, $p_1 = 20$, $p_3 = 15$ and $p_4 = 15$ psi, Eq. (1) can be rewritten in the form

$$\begin{bmatrix} 9587.4 & 0 & 0 & 0 \\ 0 & 11714.5 & 0 & 0 \\ 0 & 0 & 163.6 & 0 \\ 0 & 0 & 0 & 1963.5 \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{Bmatrix} = \begin{Bmatrix} 9587.4 * 20 \leftarrow p_1 \\ 9587.4\, p_1 + 163.6\, p_3 + 1963.5\, p_4 = 223655 \\ 163.6 * 15 \leftarrow p_3 \\ 1963.5 * 15 \leftarrow p_4 \end{Bmatrix} \quad (2)$$

Eq. (2) gives the nodal pressures as:

$p_1 = 20$ psi, $p_2 = \dfrac{223655}{11714.5} = 19.0921$ psi,

$p_3 = 15$ psi, $p_4 = 15$ psi

(b) Volume flow rates in pipe elements:

$$Q_1 = c_1 (p_1 - p_2) = 9587.4\,(20 - 19.0921) = 8704.323 \text{ in}^3/\text{sec}$$

$$Q_2 = c_2 (p_1 - p_2) = 1636.0\,(19.0921 - 15) = 669.4676 \text{ in}^3/\text{sec}$$

$$Q_3 = c_3 (P_2 - P_4) = 1963.5 (19.0921 - 15)$$
$$= 8034.838 \ \text{in}^3/\text{sec}$$

It can be verified that the inflow rate (Q_1) is equal to the total outflow rate ($Q_2 + Q_3$).

(c) Reynolds number for flow in each of the pipe elements:

$$R_e = \frac{\rho v d}{\mu} = \frac{\rho Q d}{\left(\frac{\pi d^2}{4}\right)\mu} = \frac{4 \rho Q}{\pi d \mu}$$

For pipe element 1:

$$R_e = 4 \left(\frac{1}{12}\right)^4 \frac{\rho Q}{\pi \mu d} = 4 \left(\frac{1}{12}\right)^4 \frac{(1.9)(8704.3)}{\pi (1.6 \times 10^{-6})(5)}$$
$$= 0.126936 \times 10^6$$

For pipe element 2:

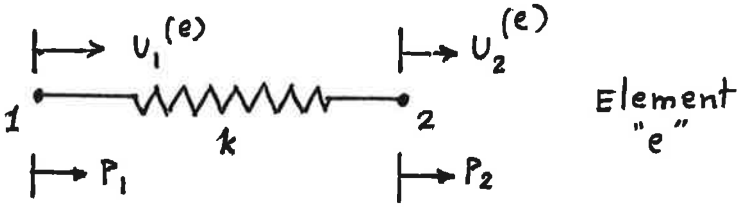
$$R_e = 4 \left(\frac{1}{12}\right)^4 \frac{(1.9)(669.5)}{\pi (1.6 \times 10^{-6})(2)} = 0.024411 \times 10^6$$

For pipe element 3:

$$R_e = 4 \left(\frac{1}{12}\right)^4 \frac{(1.9)(8034.8)}{\pi (1.6 \times 10^{-6})(4)} = 0.146465 \times 10^6$$

(d) Since $R_e > 2000$ in each pipe elements (segments), the flow is turbulent in all pipe elements.

1.12



When nodal forces P_1 and P_2 act along the nodal displacements $u_1^{(e)}$ and $u_2^{(e)}$, respectively, the strain energy of the spring is given by

$$\pi = \frac{1}{2}\,(\text{net force})\,(\text{net displacement})$$
$$= \frac{1}{2}\left\{k\left(u_2^{(e)} - u_1^{(e)}\right)\right\}\left(u_2^{(e)} - u_1^{(e)}\right)$$
$$= \frac{1}{2}\,k\left(u_2^{(e)} - u_1^{(e)}\right)^2 \qquad (1)$$

Eq.(1) can be expressed in matrix form as

$$\pi = \frac{1}{2}\left\{u_1^{(e)} \quad u_2^{(e)}\right\}\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} u_1^{(e)} \\ u_2^{(e)} \end{Bmatrix} \qquad (2)$$

or

$$\pi = \frac{1}{2}\,\vec{U}^{(e)^T}\left[K^{(e)}\right]\vec{U}^{(e)} \qquad (3)$$

where

$\vec{U}^{(e)} = \begin{Bmatrix} u_1^{(e)} \\ u_2^{(e)} \end{Bmatrix}$ = vector of nodal displacements of element

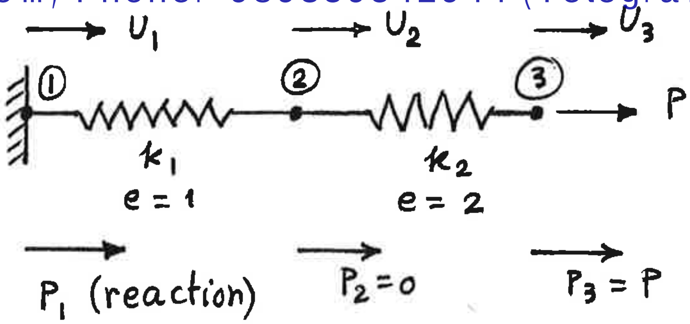
and

$[K^{(e)}]$ = stiffness matrix of element.

By comparing Eqs. (2) and (3), the stiffness matrix of the element can be identified as

$$\left[K^{(e)}\right] = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad (4)$$

1.13



Element stiffness matrices:

$$[k^{(1)}] = k_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{array}{c} \begin{array}{cc} U_1 & U_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_1 \\ U_2 \end{array} \quad N/m \qquad (1)$$

$$[k^{(2)}] = k_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 5 \times 10^5 \begin{array}{c} \begin{array}{cc} U_2 & U_3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_2 \\ U_3 \end{array} \quad N/m \qquad (2)$$

Assembled stiffness matrix of the system:

$$[\underset{\sim}{K}] = 10^5 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+5 & -5 \\ 0 & -5 & 5 \end{bmatrix} = 10^5 \begin{array}{c} \begin{array}{ccc} U_1 & U_2 & U_3 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 \\ -1 & 6 & -5 \\ 0 & -5 & 5 \end{bmatrix} \end{array} \begin{array}{c} U_1 \\ U_2 \\ U_3 \end{array} \qquad (3)$$

Equilibrium equations of the system:

$$10^5 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 6 & -5 \\ 0 & -5 & 5 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ 0 \\ 1000 \end{Bmatrix} \qquad (4)$$

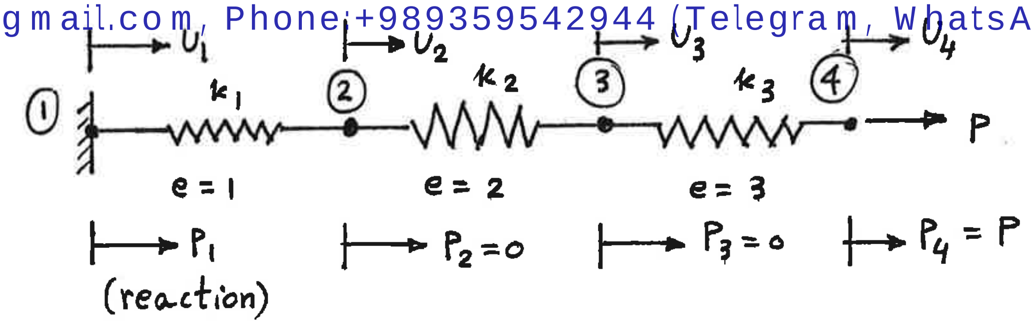
By using the condition $U_1 = 0$ in Eq. (4), i.e., by deleting the row and column corresponding to U_1 in Eq. (4), we obtain

$$10^5 \begin{bmatrix} 6 & -5 \\ -5 & 5 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1000 \end{Bmatrix} \qquad (5)$$

Solution of Eq. (5) gives the nodal displacements:

$$U_2 = 0.010 \text{ m}, \quad U_3 = 0.012 \text{ m} \qquad (6)$$

1.14



Element matrices:

$$[k^{(1)}] = k_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{array}{c} \begin{array}{cc} U_1 & U_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_1 \\ U_2 \end{array} \quad N/m$$

$$[k^{(2)}] = k_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 2 \times 10^5 \begin{array}{c} \begin{array}{cc} U_2 & U_3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_2 \\ U_3 \end{array} \quad N/m$$

$$[k^{(3)}] = k_3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 3 \times 10^5 \begin{array}{c} \begin{array}{cc} U_3 & U_4 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_3 \\ U_4 \end{array} \quad N/m$$

Assembled stiffness matrix of the system:

$$[\underset{\sim}{K}] = 10^5 \begin{array}{c} \begin{array}{cccc} U_1 & U_2 & U_3 & U_4 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1+2 & -2 & 0 \\ 0 & -2 & 2+3 & -3 \\ 0 & 0 & -3 & 0 \end{bmatrix} \end{array} \begin{array}{c} U_1 \\ U_2 \\ U_3 \\ U_4 \end{array} \quad N/m$$

Equilibrium equations of the system:

$$10^5 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 5 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ 0 \\ 0 \\ 1000 \end{Bmatrix} \qquad (1)$$

By incorporating the boundary condition $U_1 = 0$, Eq. (1) reduces to

$$10^5 \begin{bmatrix} 3 & -2 & 0 \\ -2 & 5 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1000 \end{Bmatrix} \qquad (2)$$

solution of Eq. (2): $U_2 = 10^{-2}$ m, $U_3 = 1.5 \times 10^{-2}$ m, $U_4 = 1.8333 \times 10^{-2}$ m.