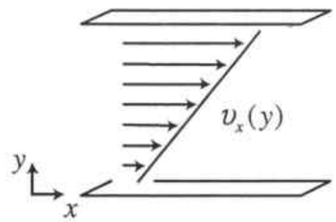
2 Solutions

Problem 2-1: In addition to internal energy, fluids carry kinetic energy. Hypothesize expressions for the advection and diffusion fluxes of kinetic energy in a unidirectional flow (such as the Couette flow illustrated, where $\upsilon_x(y)$ and $\upsilon_y = 0$). In what directions do advection and diffusion transport kinetic energy? How is the diffusion flux of kinetic energy related to the diffusion flux of momentum? Show that the diffusion flux of kinetic energy corresponds dimensionally to the rate of work performed on a unit area of the fluid.



Solution:

Kinetic energy is: $KE = \frac{1}{2} m\upsilon^2$. Here $\upsilon = \upsilon_x$, since $\upsilon_y = 0$.

On a per-volume basis $\frac{KE}{V} = \frac{1}{2}\frac{m}{V}\upsilon_x^2 = \rho\upsilon_x^2/2$.

Since $\left.\begin{matrix} advection \\ flux\ of\ X \end{matrix}\right\} = \frac{X}{V}\vec{\upsilon}$, the advection flux of KE for the illustrated flow is:

$$\left.\begin{matrix} advection \\ flux\ of\ X \end{matrix}\right\} = (\rho\upsilon_x^2/2)\upsilon_x$$

Advection transport occurs in the x-direction for the flow illustrated.

Since $\left.\begin{matrix} diffusion \\ flux\ of\ X \end{matrix}\right\} = -(coef.)\vec{\nabla}\left(\frac{X}{V}\right)$, the diffusion flux of KE for the illustrated flow is:

$$\left.\begin{matrix} diffusion \\ flux\ of\ X \end{matrix}\right\} = -\nu\frac{\partial}{\partial y}(\rho\upsilon_x^2/2)$$

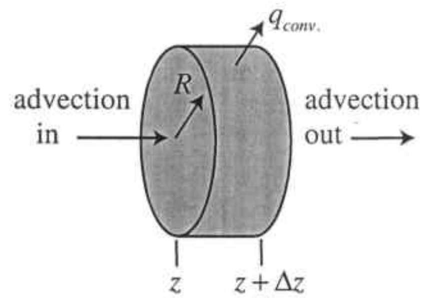
Diffusion transport occurs in the y-direction for the flow illustrated.

Notice that $\left.\begin{matrix} diffusion \\ flux\ of\ X \end{matrix}\right\} = -\nu\frac{\partial}{\partial y}(\rho\upsilon_x^2/2) = \upsilon_x \underbrace{(-\nu)\frac{\partial}{\partial y}(\rho\upsilon_x)}_{\text{diffusion flux of momentum}}$

The diffusion of kinetic energy has units of:

$$-\nu\frac{\partial}{\partial y}(\rho\upsilon_x^2/2) \rightarrow \frac{m^2}{s}\frac{1}{m}\frac{kg}{m^3}\left(\frac{m}{s}\right)^2 \rightarrow \frac{1}{s}\frac{1}{m^2}\left(\frac{kg\,m}{s^2}\right)m \rightarrow \frac{N\,m}{s\,m^2} \rightarrow \frac{(work)}{(time)(area)}$$

Problem 2-2: Express the advection of heat downstream in an incompressible pipe flow in terms of T_m and υ_m, where $\upsilon_m = \int_A \upsilon_z dA / A$ and $T_m = \int_A \upsilon_z T dA / (\upsilon_m A)$. Balance the change in advected heat with the heat lost by convection to the wall: $q_{conv.} = h(T_m - T_s)$. Find an expression for dT_m / dz.



Assuming h is not a function of z, solve for $T_m(z)$ using $T_m(0) = T_m(z=0)$ as an initial condition. Express the change in mean temperature as a function of the dimensionless quantities:

$$Nu_D = \frac{h(2R)}{k} \text{ (Nusselt number)}, \quad \Pr = \frac{\nu}{\alpha} \text{ (Prandtl number)}$$

and $$\mathrm{Re}_D = \frac{\upsilon_m(2R)}{\nu} \text{ (Reynolds number)}$$

Solution:

Heat carried by advection is: $\upsilon_z u = \upsilon_z \rho C T$, where $C_v = C_p = C$ for an incompressible flow. The total heat carried by the flow is
$\rho C \int_A \upsilon_z T dA = \rho C(\upsilon_m A) T_m$, where $A = \pi R^2$

The balance of heat between z and $z + \Delta z$ is:

$$\begin{aligned} (in) &= (out) \\ \left[\rho C(\upsilon_m A) T_m\right]\Big|_z &= \left[\rho C(\upsilon_m A) T_m\right]\Big|_{z+\Delta z} + q_{conv.}\Gamma \Delta z \end{aligned}$$

where $q_{conv.} = h(T_m - T_s)$ and $\Gamma = 2\pi R$.

Since $\left[\rho C(\upsilon_m A)\right]\Big|_z = \left[\rho C(\upsilon_m A)\right]\Big|_{z+\Delta z}$, the balance is rewritten as:

$$0 = \rho C(\upsilon_m A) \frac{T_m\big|_{z+\Delta z} - T_m\big|_z}{\Delta z} + h(T_m - T_s)\Gamma$$

Since $\lim_{\Delta z \to 0} \frac{T_m\big|_{z+\Delta z} - T_m\big|_z}{\Delta z} = \frac{dT_m}{dz}$, the heat balance becomes:

$$\frac{dT_m}{dz} = -\frac{h(T_m - T_s)\Gamma}{\rho C(\upsilon_m A)} = -2\frac{(T_m - T_s)}{R}\frac{\alpha}{\nu}\frac{h(2R)}{k}\frac{\nu}{\upsilon_m(2R)} = -2\frac{(T_m - T_s)}{R}\frac{Nu_D}{\Pr \mathrm{Re}_D}$$

Since,

$$\frac{d(T_m - T_s)}{(T_m - T_s)} = \frac{-2Nu_D}{\Pr \mathrm{Re}_D}\frac{dz}{R}$$

The streamwise change in mean temperature can be determined from:

$$\ln(T_m - T_s)\Big|_{z=0}^{z} = \frac{-2Nu_D}{\Pr \mathrm{Re}_D}\frac{z}{R} \text{ or } \ln\left(\frac{T_m(z) - T_s}{T_m(0) - T_s}\right) = \frac{-2Nu_D}{\Pr \mathrm{Re}_D}\frac{z}{R}$$

such that: $\dfrac{T_m(z) - T_s}{T_m(0) - T_s} = \exp\left(\dfrac{-2Nu_D}{\Pr \mathrm{Re}_D}\dfrac{z}{R}\right)$.

Problem 2-3: Show that the diffusion laws $c_A(\vec{v}_A - \vec{v}^*) = -c\,Đ_{AB}^{\chi}\vec{\nabla}\chi_A$ and $\rho_A(\vec{v}_A - \vec{v}) = -\rho\,Đ_{AB}^{\omega}\vec{\nabla}\omega_A$ for a binary system lead to equivalent definitions for the species diffusivities $Đ_{AB}^{\chi}$ and $Đ_{AB}^{\omega}$.

Solution:

$$\frac{\chi_A(\vec{v}_A - \vec{v}^*)}{\vec{\nabla}\chi_A} = Đ_{AB}^{\chi} = Đ_{AB}^{\omega} = \frac{\omega_A(\vec{v}_A - \vec{v})}{\vec{\nabla}\omega_A}.$$

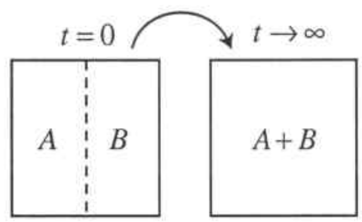
$$\frac{\chi_A(\vec{v}_A - \vec{v}^*)}{\vec{\nabla}\chi_A} = \frac{\omega_A(\vec{v}_A - \vec{v})}{\vec{\nabla}\omega_A} \quad \text{or} \quad \frac{\chi_A\chi_B(\vec{v}_A - \vec{v}_B)}{\vec{\nabla}\chi_A} = \frac{\omega_A\omega_B(\vec{v}_A - \vec{v}_B)}{\vec{\nabla}\omega_A}$$

$$\frac{\chi_A\chi_B}{\vec{\nabla}\chi_A} = \frac{\omega_A\omega_B}{\vec{\nabla}\omega_A} \quad \text{or} \quad \frac{1}{c^2}\frac{c_A c_B}{\vec{\nabla}(c_A/c)} = \frac{1}{\rho^2}\frac{\rho_A\rho_B}{\vec{\nabla}(\rho_A/\rho)} \quad \text{or} \quad \frac{\vec{\nabla}(\rho_A/\rho)}{\vec{\nabla}(c_A/c)} = \frac{\hat{M}_A\hat{M}_B}{\hat{M}^2}$$

$$\frac{\rho\vec{\nabla}\rho_A - \rho_A\vec{\nabla}\rho}{c\vec{\nabla}c_A - c_A\vec{\nabla}c} = \hat{M}_A\hat{M}_B \quad \text{or} \quad \frac{(\rho_A+\rho_B)\vec{\nabla}\rho_A - \rho_A\vec{\nabla}(\rho_A+\rho_B)}{(c_A+c_B)\vec{\nabla}c_A - c_A\vec{\nabla}(c_A+c_B)} = \hat{M}_A\hat{M}_B$$

$$\frac{\rho_B\vec{\nabla}\rho_A - \rho_A\vec{\nabla}\rho_B}{c_B\vec{\nabla}c_A - c_A\vec{\nabla}c_B} = \hat{M}_A\hat{M}_B \quad \text{or} \quad \frac{\rho_B\vec{\nabla}\rho_A - \rho_A\vec{\nabla}\rho_B}{\rho_B\vec{\nabla}\rho_A - \rho_A\vec{\nabla}\rho_B} = 1 \quad \text{which is true!}$$

Problem 2-4: Consider a container of two ideal gases, at the same temperature and pressure, separated by a partition. When the partition is removed, a concentration gradient is established in the container that drives diffusion transport. Eventually, a homogenous mixture of the two gases is formed. Demonstrate that diffusion transport occurring in this process is not reversible.



Solution:

Denoting the initial state as state 1, and the final state as state 2, the change in entropy that occurs with the change of state is given by:

$$S_2 - S_1 = m_A(s_{A,2} - s_{A,1}) + m_B(s_{B,2} - s_{B,1})$$

For an isothermal process, $ds = C_v \overbrace{dT/T}^{=0} - R d\rho/\rho$ for an ideal gas. Therefore, the change in entropy is:

$$S_2 - S_1 = -m_A R_A \ln\frac{\rho_{A,2}}{\rho_{A,1}} - m_B R_B \ln\frac{\rho_{B,2}}{\rho_{B,1}}$$

where m_A and m_B are the masses of the two gasses.

The density of each state is given by $\rho = m/\mathcal{V}$. Since the amount of mass is not changing, changes in density are simply a result of changes in volume. Therefore, the change in entropy may be expressed as:

$$S_2 - S_1 = -m_A R_A \ln\frac{\mathcal{V}_{A,1}}{\mathcal{V}_{A,2}} - m_B R_B \ln\frac{\mathcal{V}_{B,1}}{\mathcal{V}_{B,2}}$$

Since $\ln\left(\mathcal{V}_{A,1}/\mathcal{V}_{A,2}\right) < 0$ and $\ln\left(\mathcal{V}_{B,1}/\mathcal{V}_{B,2}\right) < 0$, one can conclude that the diffusion process that leads to state 2 requires that $S_2 - S_1 > 0$. Since not transport of entropy has occurred, the rise in entropy of the system must be the result of entropy generation. Therefore, this process cannot be reversed without violating the second law of thermodynamics.