1. VECTORS, MATRICES AND APPLICATIONS

1.1. Vectors

1. By the Triangle Inequality, $||\mathbf{v}+\mathbf{w}|| \le ||\mathbf{v}|| + ||\mathbf{w}||$ for two vectors $\mathbf{v}$ and $\mathbf{w}$. Assume that $\mathbf{v}$ and $\mathbf{w}$ are parallel and have the same direction, i.e., $\mathbf{w} = t\mathbf{v}$ and $t > 0$. In that case,

$$||\mathbf{v}+\mathbf{w}|| = ||\mathbf{v}+t\mathbf{v}|| = ||(1+t)\mathbf{v}|| = |1+t|\,||\mathbf{v}|| = (1+t)\,||\mathbf{v}||$$

(since $t > 0$, it follows that $1+t > 0$ and so $|1+t| = 1+t$) and

$$||\mathbf{v}|| + ||\mathbf{w}|| = ||\mathbf{v}|| + ||t\mathbf{v}|| = ||\mathbf{v}|| + |t|\,||\mathbf{v}|| = (1+t)||\mathbf{v}||$$

(since $t > 0$, $|t| = t$). Therefore, to illustrate $||\mathbf{v}+\mathbf{w}|| = ||\mathbf{v}|| + ||\mathbf{w}||$ we can choose any pair of parallel vectors with the same direction. For example, $\mathbf{v} = \mathbf{i}$ and $\mathbf{w} = 4\mathbf{i}$.

If $\mathbf{v}$ and $\mathbf{w}$ are non-parallel, then they form a triangle with sides equal to $||\mathbf{v}||$, $||\mathbf{w}||$ and $||\mathbf{v}+\mathbf{w}||$, in which case $||\mathbf{v}+\mathbf{w}|| < ||\mathbf{v}|| + ||\mathbf{w}||$. As an example, take $\mathbf{v} = \mathbf{i}$ and $\mathbf{w} = \mathbf{i}+\mathbf{j}$.

Notice that as the angle between two vectors (whose magnitudes are fixed) gets larger and larger, the magnitude of their sum gets smaller and smaller. Therefore, to find $\mathbf{w}$, $\mathbf{v}$ such that $||\mathbf{v}+\mathbf{w}|| < (||\mathbf{v}||+||\mathbf{w}||)/2$ we can look for vectors that form a larger angle, like $\mathbf{v} = -\mathbf{i}+\mathbf{j}$ and $\mathbf{w} = \mathbf{i}$. In this case, $||\mathbf{v}+\mathbf{w}|| = ||\mathbf{j}|| = 1$ and $(||\mathbf{v}||+||\mathbf{w}||)/2 = (\sqrt{2}+1)/2 > 1$.

2. Let $\mathbf{v} = (v_1, v_2, v_3)$ be any vector in $\mathbb{R}^3$. Then $\alpha\mathbf{v} = (\alpha v_1, \alpha v_2, \alpha v_3)$, for $\alpha \in \mathbb{R}$, and the definition of the length of a vector gives

$$||\alpha\mathbf{v}|| = \sqrt{(\alpha v_1)^2 + (\alpha v_2)^2 + (\alpha v_3)^2} = \sqrt{\alpha^2(v_1^2+v_2^2+v_3^2)} = |\alpha|\sqrt{v_1^2+v_2^2+v_3^2} = |\alpha|||\mathbf{v}||.$$

3. Assume that $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ (same proof works in any dimension). If $||\mathbf{v}|| = 0$ then $\sqrt{v_1^2+v_2^2} = 0$ and $v_1^2+v_2^2 = 0$. Therefore, $v_1 = v_2 = 0$ and $\mathbf{v} = \mathbf{0}$. This proves it one way. The other implication is immediate: if $\mathbf{v} = \mathbf{0} = (0,0)$, then $||\mathbf{v}|| = \sqrt{0^2+0^2} = 0$.

4. The values $r = 0$ and $\theta = \pi/2$ represent the pole (recall that the pole can be represented as $(0,\theta)$ for $0 \le \theta < 2\pi$). Alternatively, $x = r\cos\theta = 0\cos(\pi/2) = 0$ and $y = r\sin\theta = 0\sin(\pi/2) = 0$; so $(0,0)$ are Cartesian coordinates of a point whose polar coordinates are $(0,\pi/2)$. For the point whose polar coordinates are $(10,\pi/2)$, we get $x = r\cos\theta = 10\cos(\pi/2) = 0$ and $y = r\sin\theta = 10\sin(\pi/2) = 10$; therefore, $(10,\pi/2)$ is represented as $(0,10)$. For $(2, 3\pi/4)$, we get $x = 2\cos(3\pi/4) = -\sqrt{2}$ and $y = 2\sin(3\pi/4) = \sqrt{2}$; so its Cartesian coordinates are $(-\sqrt{2}, \sqrt{2})$. Similarly, for $(1,\pi/6)$ we get $x = 1\cos(\pi/6) = \sqrt{3}/2$ and $y = 1\sin(\pi/6) = 1/2$ and so its Cartesian coordinates are $(\sqrt{3}/2, 1/2)$. Finally, for $(12, 3\pi/2)$ we get $x = 12\cos(3\pi/2) = 0$ and $y = 12\sin(3\pi/2) = -12$; i.e., its Cartesian coordinates are $(0,-12)$.

5. If $x = y = 2$, then $r = \sqrt{x^2+y^2} = \sqrt{8}$ and $\tan\theta = 1$. Hence $\theta = \pi/4$ and the point $(2,2)$ is represented in polar coordinates as $(\sqrt{8}, \pi/4)$. For the remaining three points, $r = \sqrt{8}$ as well. For $(-2,2)$ we get $\tan\theta = 2/(-2) = -1$, i.e., $\theta = -\pi/4 + k\pi$ (k is an integer). Since the point $(-2,2)$ is in the second quadrant, $\theta = 3\pi/4$, and $(\sqrt{8}, 3\pi/4)$ are its polar coordinates. Similarly, for the point $(2,-2)$ we get $\tan\theta = -1$ and (the point is in the fourth quadrant) $\theta = 7\pi/4$. Hence $(\sqrt{8}, 7\pi/4)$ are its polar coordinates. Finally, from $x=-2$ and $y=-2$ we get $\tan\theta = 1$ and (we are now in the third quadrant) $\theta = 5\pi/4$. Consequently, $(\sqrt{8}, 5\pi/4)$ are the polar coordinates of $(-2,-2)$.

6. For the point whose Cartesian coordinates are $(0,-3)$ we get $r = \sqrt{x^2+y^2} = 3$ and $\theta = \arctan(y/x) = \arctan(-3/0)$, i.e., $\theta = \pi/2 + k\pi$ (k is an integer). Since the point is on the negative y-axis, it follows that $\theta = 3\pi/2$. Therefore, the point $(-3,0)$ has polar coordinates $(3, 3\pi/2)$. For the point $(1,\sqrt{3})$ we get $r = \sqrt{1+3} = 2$ and $\theta = \arctan(\sqrt{3}/1) = \pi/3$, since we are in the first quadrant. It follows that $(1,\sqrt{3})$ is represented in polar coordinates as $(2,\pi/3)$. Similarly, for $(-\sqrt{3},-1)$ we get $r = \sqrt{1+3} = 2$ and $\theta = \arctan(-1/-\sqrt{3}) = 7\pi/6$ (since $(-\sqrt{3},-1)$ lies in the third quadrant); so $(2, 7\pi/6)$ are the corresponding polar coordinates. Finally, $(-2,0)$ lies in the second quadrant, so from $\arctan(0/-2) = k\pi$ we get $\theta = \pi$; since $r=2$ it follows that the polar coordinates of $(-2,0)$ are $(2,\pi)$.

7. Rewrite the Triangle Inequality as $||\mathbf{a}+\mathbf{b}|| \le ||\mathbf{a}|| + ||\mathbf{b}||$ and substitute $\mathbf{a} = \mathbf{v} - \mathbf{w}$ and $\mathbf{b} = \mathbf{w}$. Thus $||\mathbf{v}-\mathbf{w}+\mathbf{w}|| \le ||\mathbf{v}-\mathbf{w}|| + ||\mathbf{w}||$, so $||\mathbf{v}|| \le ||\mathbf{v}-\mathbf{w}|| + ||\mathbf{w}||$ and $||\mathbf{v}-\mathbf{w}|| \ge ||\mathbf{v}|| - ||\mathbf{w}||$. Consider a triangle whose sides are $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$. The above inequality states that the difference of lengths of two sides in a triangle is smaller than the length of the third side. The equality holds in the case when $\mathbf{v}$ and $\mathbf{w}$ are parallel, of the same direction, and $||\mathbf{v}|| \ge ||\mathbf{w}||$.

8. Exercise 7 tells us that $||\mathbf{v}-\mathbf{w}|| \ge ||\mathbf{v}|| - ||\mathbf{w}||$; the equality holds when $\mathbf{v}$ and $\mathbf{w}$ are parallel, of the same direction and are such that $||\mathbf{v}|| \ge ||\mathbf{w}||$. So take, for example, $\mathbf{v} = 5\mathbf{i}$ and $\mathbf{w} = 2\mathbf{i}$.

9. If $\mathbf{v}$ and $\mathbf{w}$ are parallel of opposite directions, then $\mathbf{v}+\mathbf{w} = \mathbf{0}$ and hence 0 is the smallest value of $||\mathbf{v}+\mathbf{w}||$. By the Triangle Inequality $||\mathbf{v}+\mathbf{w}|| \le ||\mathbf{v}|| + ||\mathbf{w}||$, the largest value of $||\mathbf{v}+\mathbf{w}||$ is $||\mathbf{v}|| + ||\mathbf{w}|| = 2$ (and this happens when $\mathbf{v}$ and $\mathbf{w}$ are parallel and of the same direction; i.e., when $\mathbf{v} = \mathbf{w}$, since both vectors are of the same length).

10. Let $P(x,y)$ be a point on the ellipse and $A_1(-e,0)$ and $A_2(e,0)$ the given points. From $d(A_1,P) + d(A_2,P) = \epsilon$ it follows that $\sqrt{(x+e)^2+y^2} + \sqrt{(x-e)^2+y^2} = \epsilon$ and

$$(x+e)^2 + y^2 + (x-e)^2 + y^2 + 2\sqrt{(x+e)^2+y^2}\sqrt{(x-e)^2+y^2} = \epsilon^2.$$

Simplifying, we get

$$2x^2 + 2y^2 + 2e^2 + 2\sqrt{(x+e)^2+y^2}\sqrt{(x-e)^2+y^2} = \epsilon^2$$

and

$$\sqrt{x^2+y^2+e^2+2xe}\sqrt{x^2+y^2+e^2-2xe} = \epsilon^2/2 - (x^2+y^2+e^2).$$

Square both sides to get

$$(x^2+y^2+e^2)^2 - (2xe)^2 = \epsilon^4/4 + (x^2+y^2+e^2)^2 - \epsilon^2(x^2+y^2+e^2);$$

hence $-4x^2e^2 + \epsilon^2x^2 + \epsilon^2y^2 = \epsilon^4/4 - \epsilon^2e^2$ and

$$x^2(-4e^2 + \epsilon^2) + \epsilon^2y^2 = \epsilon^2(1/4)(\epsilon^2 - 4e^2).$$

It follows that

$$\frac{x^2}{\epsilon^2/4} + \frac{y^2}{\epsilon^2/4 - e^2} = 1.$$

11. By definition, $||\mathbf{v}|| = \sqrt{0^2 + 2^2 + (-1)^2} = \sqrt{5}$.

12. By definition, $||\mathbf{v}|| = \sqrt{\sin^2\theta + \cos^2\theta + 1} = \sqrt{2}$.

13. Since $||\mathbf{w}|| = \sqrt{9+16} = 5$, it follows that

$$\mathbf{v} = (3/5)\mathbf{i} + (4/5)\mathbf{j} \qquad \text{and} \qquad ||\mathbf{v}|| = \sqrt{9/25 + 16/25} = 1.$$

In general, $\mathbf{v} = \mathbf{w}/||\mathbf{w}|| = (1/||\mathbf{w}||)\mathbf{w}$ has the same direction as $\mathbf{w}$. Its length is 1 (it is called the unit vector in the direction of $\mathbf{w}$), since

$$||\mathbf{v}|| = \left|\left|\frac{\mathbf{w}}{||\mathbf{w}||}\right|\right| = \frac{1}{||\mathbf{w}||}||\mathbf{w}|| = 1.$$

14. By definition, $||\mathbf{w}||^2 = 1^2 + (-1)^2 + 2^2 = 6$. It follows that $\mathbf{v} = \frac{1}{6}\mathbf{i} - \frac{1}{6}\mathbf{j} + \frac{2}{6}\mathbf{k}$ and $||\mathbf{v}|| = \sqrt{1/36 + 1/36 + 4/36} = 1/\sqrt{6}$.

15. Let $A(a_1, a_2)$, $B(b_1, b_2)$ be points in $\mathbb{R}^2$. If A and B lie on the same horizontal line, then their distance is $|b_1 - a_1|$. Since, in that case, $a_2 = b_2$, we can write (recall that $|x| = \sqrt{x^2}$)

$$|b_1 - a_1| = \sqrt{(b_1 - a_1)^2} = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}.$$

If A and B lie on the same vertical line, then their distance is $|b_2 - a_2|$; since $a_1 = b_1$, we obtain the same formula. If A and B do not lie on the same horizontal or the same vertical line, then construct the triangle as shown in Figure 15(a). It is a right triangle with sides $|b_1 - a_1|$ and $|b_2 - a_2|$. By the Pythagorean Theorem, its hypotenuse is $d(A, B) = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}$.

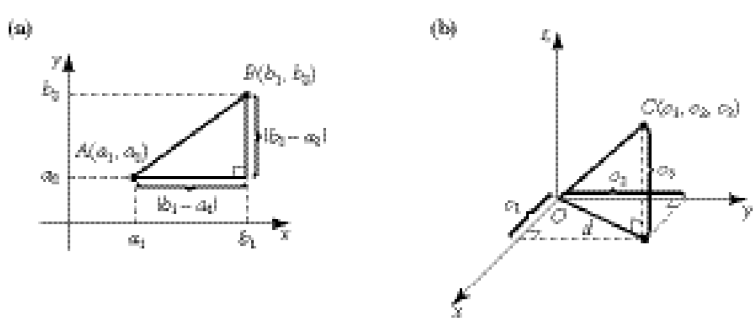

Figure 15

The proof in $\mathbb{R}^3$ is done in a similar way. Here is an alternative: first, compute the distance $d(O, C)$ from a point $C(c_1, c_2, c_3)$ to the origin, see Figure 15(b). By the Pythagorean

Theorem, $d^2 = c_1^2 + c_2^2$ and (again by the Pythagorean Theorem) $d(O,C) = \sqrt{d^2 + c_3^2} = \sqrt{c_1^2 + c_2^2 + c_3^2}$. Now take any points $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ in $\mathbb{R}^3$. Take a line segment $\overline{AB}$ and translate it by the vector $-(a_1, a_2, a_3)$, thus obtaining the segment $\overline{OC}$, joining the origin O and the point $C(b_1 - a_1, b_2 - a_2, b_3 - a_3)$. By the formula we have just derived, the length of $\overline{AB}$ = distance from the origin to $C = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$.

16. Triangle Law: Let $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$. If $\overrightarrow{OA}$ is the directed line segment that starts at O, then A has coordinates (v_1, v_2). Choose the representative directed line segment of $\mathbf{w} = (w_1, w_2)$ that starts at $A(v_1, v_2)$, and call it $\overrightarrow{AB}$. Then B has coordinates $(v_1 + w_1, v_2 + w_2)$. Consequently, the vector represented by $\overrightarrow{OB}$ (i.e., the third side of the triangle OAB) has coordinates $(v_1 + w_1, v_2 + w_2)$; and these are the coordinates of $\mathbf{v} + \mathbf{w}$.

Parallelogram Law: Let $\mathbf{v}$ and $\mathbf{w}$ be represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively. By drawing parallels to $\overline{OA}$ and $\overline{OB}$ we obtain the parallelogram $OACB$, as shown in Figure 16. We have to find the coordinates of C. Since the triangles $OB'B$ and $AC_1'C$ are congruent (they are similar and $d(O,B) = d(A,C)$) it follows that $d(A', C_1) = d(A, C_1') = d(O, B')$ and hence $d(O, C_1) = d(O, A') + d(A', C_1) = v_1 + w_1$. Similarly, $d(O, C_2) = v_2 + w_2$. Therefore, the vector represented by $\overrightarrow{OC}$ has components $(v_1 + w_1, v_2 + w_2)$ and is thus equal to $\mathbf{v} + \mathbf{w}$.

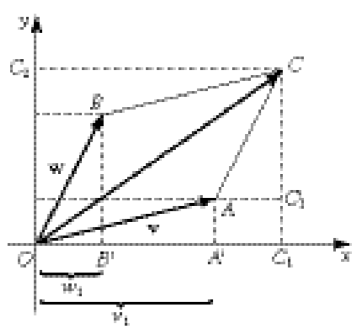

Figure 16

17. Recall that the representative of a vector $\mathbf{v} = (v_1, v_2, v_3)$ whose tail is at $A(a_1, a_2, a_3)$ is the directed line segment $\overrightarrow{AB}$, where the point B has coordinates $B(a_1 + v_1, a_2 + v_2, a_3 + v_3)$. Let $\mathbf{v} = (0, 2, -1)$. The directed line segment $\overrightarrow{A_1B_1}$, where $A_1(0,1,1)$ and $B_1(0,3,0)$ is the representative that starts at $A_1(0,1,1)$. Similarly, the directed line segments $\overrightarrow{A_2B_2}$, $\overrightarrow{A_3B_3}$ and $\overrightarrow{A_4B_4}$, where $A_2(0,3,0)$, $B_2(0,5,-1)$, $A_3(8,9,-4)$, $B_3(8,11,-5)$, $A_4(10,-1,4)$ and $B_4(10,1,3)$, are the representatives of $\mathbf{v}$ that start at A_2, A_3 and A_4 respectively.

18. The vector represented by the directed line segment $\overrightarrow{AB}$, where $A(3,4)$ and $B(-1,0)$ is $\mathbf{v} = (-1-3, 0-4) = (-4,-4)$. The representative of $\mathbf{v} = (-4,-4)$ whose tail is at $A_1(0,2)$ is the directed line segment $\overrightarrow{A_1B_1}$, where $B_1 = (0-4, 2-4) = (-4,-2)$. Similarly, the directed line segments $\overrightarrow{A_2B_2}$, and $\overrightarrow{A_3B_3}$, where $A_2(1,1)$, $B_2(-3,-3)$, $A_3(-4,-2)$ and $B_3(-8,-6)$ are the representatives of $\mathbf{v}$ that start at A_2 and A_3 respectively.

19. We use elementary vector operations. First of all, $\mathbf{a}-2\mathbf{b} = (2\mathbf{i}-\mathbf{j}+\mathbf{k})-2(\mathbf{k}-3\mathbf{i}) = 8\mathbf{i}-\mathbf{j}-\mathbf{k}$. Since $\|\mathbf{c}\| = 2$, it follows that $\mathbf{a} - \mathbf{c}/\|\mathbf{c}\| = (2\mathbf{i}-\mathbf{j}+\mathbf{k}) - 2\mathbf{i}/2 = \mathbf{i}-\mathbf{j}+\mathbf{k}$. Similarly, $3\mathbf{a}+\mathbf{c}-\mathbf{j}+\mathbf{k} = 3(2\mathbf{i}-\mathbf{j}+\mathbf{k})+2\mathbf{i}-\mathbf{j}+\mathbf{k} = 8\mathbf{i}-4\mathbf{j}+4\mathbf{k}$. Since $\mathbf{b}+2\mathbf{a} = (\mathbf{k}-3\mathbf{i})+2(2\mathbf{i}-\mathbf{j}+\mathbf{k}) = \mathbf{i}-2\mathbf{j}+3\mathbf{k}$ and $\|\mathbf{b}+2\mathbf{a}\| = \sqrt{14}$, it follows that the unit vector in the direction of $\mathbf{b}+2\mathbf{a}$ is $(\mathbf{b}+2\mathbf{a})/\|\mathbf{b}+2\mathbf{a}\| = (\mathbf{i}-2\mathbf{j}+3\mathbf{k})/\sqrt{14}$.

20. Using the polar form of a vector, we get $\mathbf{v} = \|\mathbf{v}\|(\cos\theta\mathbf{i}+\sin\theta\mathbf{j}) = 10(\cos(2\pi/3)\mathbf{i} + \sin(2\pi/3)\mathbf{j}) = -5\mathbf{i}+5\sqrt{3}\mathbf{j}$.

21. The vector $-3\mathbf{i}$ has length 3 and it makes an angle of π radians with respect to the positive x-axis. Consequently, its polar form is $-3\mathbf{i} = 3(\cos\pi\,\mathbf{i}+\sin\pi\,\mathbf{j})$. The length of $\mathbf{i}/2-\mathbf{j}$ is $\sqrt{5}/2 \approx 1.11803$. We need the angle between the positive x-axis and $\mathbf{i}/2-\mathbf{j}$ (notice that $\mathbf{i}/2-\mathbf{j}$ is in the fourth quadrant). From $\tan\theta_1 = -1/(1/2) = -2$ it follows that $\theta_1 = \arctan(-2)$, and the required angle is $2\pi + \arctan(-2) \approx 5.17604$ rad. Hence $\mathbf{i}/2-\mathbf{j} \approx 1.11803(\cos 5.17604\,\mathbf{i} + \sin 5.17604\,\mathbf{j})$. Finally, let $\mathbf{v} = \mathbf{i}-4\mathbf{j}$ ($\mathbf{v}$ is in the fourth quadrant). Then $\|\mathbf{v}\| = \sqrt{17} \approx 4.12311$ and $\tan\theta_2 = -4$, so that the angle between the positive x-axis and the vector $\mathbf{v}$ is $2\pi+\arctan(-4) \approx 4.95737$ rad. Therefore, $\mathbf{i}-4\mathbf{j} \approx 4.12311(\cos 4.95737\,\mathbf{i} + \sin 4.95737\,\mathbf{j})$.

22. Write $\mathbf{a} = 2\mathbf{i}+\mathbf{j} = (2,1,0)$ and $\mathbf{b} = -\mathbf{j}-2\mathbf{k} = (0,-1,-2)$. We have to find a vector $\mathbf{x} = (x_1,x_2,x_3)$ such that $\mathbf{a}+2(\mathbf{x}-\mathbf{b}) = 3\mathbf{x}+2(\mathbf{a}-\mathbf{b})$. Expressing this equation in coordinates, we get $(2,1,0)+2(x_1-0,x_2+1,x_3+2) = (2x_1+2, 2x_2+3, 2x_3+4)$ for its left side and $3(x_1,x_2,x_3)+2(2-0,1+1,0+2) = (3x_1+4, 3x_2+4, 3x_3+4)$ for its right side. Consequently, $2x_1+2 = 3x_1+4$, $2x_2+3 = 3x_2+4$ and $2x_3+4 = 3x_3+4$, and so $x_1 = -2$, $x_2 = -1$ and $x_3 = 0$; i.e., $\mathbf{x} = (-2,-1,0)$.

23. Let $\mathbf{v} = (v_1,v_2,v_3)$. Since $(\alpha+\beta)\mathbf{v} = (\alpha+\beta)(v_1,v_2,v_3) = ((\alpha+\beta)v_1, (\alpha+\beta)v_2, (\alpha+\beta)v_3)$ and

$$\begin{aligned}\alpha\mathbf{v}+\beta\mathbf{v} &= \alpha(v_1,v_2,v_3)+\beta(v_1,v_2,v_3) = (\alpha v_1,\alpha v_2,\alpha v_3)+(\beta v_1,\beta v_2,\beta v_3)\\ &= (\alpha v_1+\beta v_1, \alpha v_2+\beta v_2, \alpha v_3+\beta v_3),\end{aligned}$$

it follows (by the distributivity of multiplication of real numbers) that $(\alpha+\beta)\mathbf{v} = \alpha\mathbf{v}+\beta\mathbf{v}$. Similarly, $(\alpha\beta)\mathbf{v} = (\alpha\beta)(v_1,v_2,v_3) = ((\alpha\beta)v_1, (\alpha\beta)v_2, (\alpha\beta)v_3)$ and

$$\alpha(\beta\mathbf{v}) = \alpha(\beta(v_1,v_2,v_3)) = \alpha(\beta v_1,\beta v_2,\beta v_3) = (\alpha(\beta v_1), \alpha(\beta v_2), \alpha(\beta v_3)),$$

together with the associativity of multiplication of real numbers prove the second identity.

24. The difference $\mathbf{v}-\mathbf{w}$ of $\mathbf{v}$ and $\mathbf{w}$ can be defined as the sum $\mathbf{v}-\mathbf{w} = \mathbf{v}+(-\mathbf{w})$. Assume that $\mathbf{v},\mathbf{w} \in \mathbb{R}^2$ — the case of $\mathbb{R}^3$ is analogous. If $\mathbf{w}$ is represented by the directed line segment $\overrightarrow{AB}$; i.e., if $\mathbf{w} = (w_1,w_2) = (b_1-a_1, b_2-a_2)$, where $A(a_1,a_2)$ and $B(b_1,b_2)$, then $-\mathbf{w} = (-w_1,-w_2) = (a_1-b_1, a_2-b_2)$. Consequently, $-\mathbf{w}$ is represented by the directed line segment $\overrightarrow{BA}$. In words, to get $-\mathbf{w}$ from $\mathbf{w}$ we have to reverse the direction: the tail of $-\mathbf{w}$ is the tip of $\mathbf{w}$ and the tip of $-\mathbf{w}$ is the tail of $\mathbf{w}$. Once we have $-\mathbf{w}$, we either use the Triangle Law or the Parallelogram Law to construct $\mathbf{v}+(-\mathbf{w})$.

1.2. Applications in Geometry and Physics

1. The equation of the line is

$$\ell(t) = (1,3) + t\,(3\mathbf{v}) = (1,3) + t(-3,-15) = (1-3t, 3-15t),$$

where $t \in \mathbb{R}$. If we replace $3\mathbf{v}$ by a vector $m\mathbf{v}$ (where $m \neq 0$; clearly, $m\mathbf{v}$ is parallel to $\mathbf{v}$), we get infinitely many parametrizations, one for each non-zero value of m:

$$\ell(t) = (1,3) + t\,m(-1,-5) = (1-tm, 3-5tm),$$

where $t \in \mathbb{R}$.

2. The equation of the line is $\ell(t) = \mathbf{a}+t\mathbf{v}$, where $\mathbf{a} = (3,2,0)$ and $\mathbf{v}$ is the vector from $(3,2,0)$ to $(0,-1,-1)$; i.e., $\mathbf{v} = (-3,-3,-1)$. Hence $\ell(t) = (3,2,0)+t(-3,-3,-1) = (3-3t, 2-3t, -t)$, where $t \in \mathbb{R}$.

3. The equation of the line is $\ell(t) = \mathbf{a} + t\mathbf{v}$, where $\mathbf{a} = (1,1)$ and $\mathbf{v}$ is the vector from $(1,1)$ to $(-2,4)$; i.e., $\mathbf{v} = (-3,3)$. Hence $\ell(t) = (1,1) + t(-3,3) = (1-3t, 1+3t)$, where $t \in \mathbb{R}$, parametrizes the given line.

To describe the half-line starting at $(1,1)$ in the direction towards $(-2,4)$, we need all vectors that point in the same direction as $\mathbf{v}$. Hence $\ell(t) = (1-3t, 1+3t)$ with $t \geq 0$ is its parametric representation.

When $t = 0$, $\ell(0) = (1,1)$; for $t = 1$, we get $\ell(1) = (-2,4)$. If $0 < t < 1$, then $\ell(t)$ is a point on the line between $(1,1)$ and $(-2,4)$. Consequently $\ell(t) = (1-3t, 1+3t)$, where $0 \leq t \leq 1$, describes the desired line segment.

4. The equation of the line is $\ell(t) = \mathbf{a} + t\mathbf{v}$, where $\mathbf{a} = (2,-2,0)$ and $\mathbf{v}$ is the vector from $(2,-2,0)$ to $(1,1,4)$; i.e., $\mathbf{v} = (-1,3,4)$. Hence $\ell(t) = (2,-2,0) + t(-1,3,4) = (2-t, -2+3t, 4t)$, $t \in \mathbb{R}$. When $t = 0$, $\ell(0) = (2,-2,0)$, and when $t = 1$, $\ell(1) = (1,1,4)$. To get all points on ℓ that lie between $(2,-2,0)$ and $(1,1,4)$ we have to take $0 \leq t \leq 1$. Hence $\overline{\ell}(t) = (2-t, -2+3t, 4t)$, $0 \leq t \leq 1$, describes the given line segment.

5. Choose $\mathbf{a} = (2,1)$ and let $\mathbf{v}$ be the vector from $(2,1)$ to $(-1,5)$; i.e., $\mathbf{v} = (-3,4)$. Then $\ell(t) = (2,1) + t(-3,4)$, $t \in \mathbb{R}$, is a parametric equation of the line containing the given points. The half-line ℓ' "contains" all vectors parallel to $\mathbf{v}$ that are of the same direction as $\mathbf{v}$. Hence $\ell'(t) = (2,1) + t(-3,4)$ with $t \geq 0$ parametrizes ℓ'.

6. Let $\mathbf{a} = (1,-1,4)$. Take $\mathbf{v}$ to be the vector from $(1,-1,4)$ to $(2,1,-3)$ (hence $\mathbf{v} = (1,2,-7)$) and let $\mathbf{w}$ be the vector from $(1,-1,4)$ to $(0,1,0)$ (hence $\mathbf{w} = (-1,2,-4)$). From (1.4) in the text it follows that the parametric equations

$$\mathbf{p} = \mathbf{a} + t\mathbf{v} + s\mathbf{w} = (1+t-s, -1+2t+2s, 4-7t-4s),$$

for $s, t \in \mathbb{R}$, represent the given plane.

To obtain the equation in x, y and z we have to eliminate the parameters t and s from $x = 1 + t - s$, $y = -1 + 2t + 2s$ and $z = 4 - 7t - 4s$. From the first and the third equations we get $4x - z = 4(1 + t - s) - (4 - 7t - 4s) = 11t$. Multiply the equation for x by 11 and replace $11t$ by $4x - z$: so $11x = 11 + 11t - 11s = 11 + (4x - z) - 11s$; it follows that $11s = 11 - 7x - z$. Multiply the equation for y by 11, and substitute the values for t and s: hence $11y = -11 + 22t + 22s = -11 + (8x - 2z) + (22 - 14x - 2z)$. Simplify to get $11y = -4z - 6x + 11$, i.e., $z = (-6x - 11y + 11)/4$.

We can generate an infinite number of parametric equations by using non-zero multiples of the given vectors $\mathbf{v}$ and $\mathbf{w}$ in equation (1.4).

7. The rectangle in question is spanned by $\mathbf{v} = 3\mathbf{i}$ and $\mathbf{w} = \mathbf{j}$; see Figure 7.

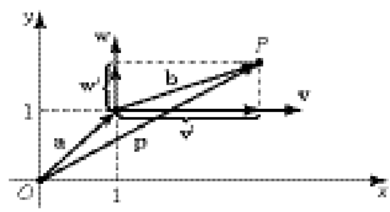

Figure 7

Let $P(x, y)$ be a point in the rectangle, $\mathbf{b}$ the vector from $(1, 1)$ to P and $\mathbf{a}$ the vector from the origin to $(1, 1)$. Then ($\mathbf{p}$ is the position vector of P) $\mathbf{p} = \mathbf{a} + \mathbf{b}$, where $\mathbf{b} = \mathbf{v}' + \mathbf{w}'$. Since P belongs to the rectangle, $\mathbf{v}' = \alpha\mathbf{v}$ and $0 \le \alpha \le 1$. Similarly, $\mathbf{w}' = \beta\mathbf{w}$ and $0 \le \beta \le 1$. It follows that $\mathbf{b} = \alpha(3\mathbf{i}) + \beta\mathbf{j}$, and therefore $\mathbf{p} = \mathbf{i} + \mathbf{j} + 3\alpha\mathbf{i} + \beta\mathbf{j} = (1 + 3\alpha)\mathbf{i} + (1 + \beta)\mathbf{j}$, $0 \le \alpha, \beta \le 1$, is a vector description of the rectangle.

8. The parametric equation of the plane spanned by $\mathbf{v}$ and $\mathbf{w}$ is (apply (1.4) with $\mathbf{a} = \mathbf{0}$) $\mathbf{p} = t\mathbf{v} + s\mathbf{w} = (t - 2s, 4t, 2t + 3s)$, $t, s \in \mathbb{R}$. A point that belongs to the given parallelogram can be expressed as $\mathbf{v}' + \mathbf{w}'$ (see Figure 1.9 in the text). Since $\mathbf{v}' = t\mathbf{v}$ and $||\mathbf{v}'|| \le ||\mathbf{v}||$, we must take $t \le 1$ (together with $t \ge 0$). Similarly, $\mathbf{w}' = s\mathbf{w}$, where $0 \le s \le 1$. It follows that $\mathbf{p} = (t - 2s, 4t, 2t + 3s)$, $0 \le s, t \le 1$, describes the given parallelogram.

9. The relative position of the two cars is either of the two differences of the displacement vectors, $\mathbf{v} = \mathbf{i} + 3\mathbf{j} - \mathbf{k} - (\mathbf{j} + 2\mathbf{k}) = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ or $-\mathbf{v}$. The distance between the cars is $||\mathbf{v}|| = \sqrt{14}$.

10. The particle will move $12 \cdot 60 = 720$ units in one minute. The (unit) direction of motion is $(\mathbf{i}+2\mathbf{j}-2\mathbf{k})/\sqrt{1^2 + 2^2 + (-2)^2} = (\mathbf{i}+2\mathbf{j}-2\mathbf{k})/3$. Therefore, the *relative* position of the particle *with respect to its initial position* after one minute is $720(\mathbf{i}+2\mathbf{j}-2\mathbf{k})/3 = 240\mathbf{i}+480\mathbf{j}-480\mathbf{k}$.

11. The particle moves along the line

$$\ell(t) = \textbf{initial point} + \textbf{displacement} = (3, 2, 4) + t(3, 0, -1),$$

$t \geq 0$. It will cross the xy-plane when the z-coordinate of $\ell(t)$ is zero; i.e., when $4 - t = 0$ or $t = 4$. The point where it crosses the plane is $\ell(4) = (15, 2, 0)$.

12. The location of the particle at time t is given by $\ell(t) = \mathbf{a} + t\mathbf{v}$, where $\mathbf{a}$ is the initial location and $\mathbf{v}$ is the velocity vector. In this case, $\mathbf{a} = (-2, 1)$ and $\mathbf{v} = ||\mathbf{v}||(\cos(\pi/3), \sin(\pi/3)) = 3(1/2, \sqrt{3}/2)$; i.e., $\ell(t) = (-2, 1) + 3t(1/2, \sqrt{3}/2)$. The location after 10 seconds is $\ell(10) = (-2, 1) + 3(10)(1/2, \sqrt{3}/2) = (13, 1 + 15\sqrt{3})$. The velocity is (constant throughout the motion and equal to) $3(1/2, \sqrt{3}/2)$.

13. The information given describes the vector $\mathbf{F}$ in polar form. Hence

$$\mathbf{F} = ||\mathbf{F}||(\cos(\pi/6)\mathbf{i} + \sin(\pi/6)\mathbf{j}) = 10\left(\tfrac{\sqrt{3}}{2}\mathbf{i} + \tfrac{1}{2}\mathbf{j}\right) = 5\sqrt{3}\mathbf{i} + 5\mathbf{j}.$$

14. Place the coordinate axes so that the x-axis points east and the y-axis points north. Then $\mathbf{v}_A = 100\mathbf{j}$ and $\mathbf{v}_B = 80\mathbf{i}/\sqrt{2} + 80\mathbf{j}/\sqrt{2}$ (since northeastern direction is represented by the (unit) vector $\mathbf{i}/\sqrt{2} + \mathbf{j}/\sqrt{2}$). It follows that the relative velocity of B as seen by A is $\mathbf{v}_B - \mathbf{v}_A = (80\mathbf{i}/\sqrt{2} + 80\mathbf{j}/\sqrt{2}) - 100\mathbf{j} = (80/\sqrt{2})\mathbf{i} + (80/\sqrt{2} - 100)\mathbf{j}$.

15. The position vectors of the four masses are $\mathbf{0}$, $\mathbf{i} - 2\mathbf{j}$, $\mathbf{i} + \mathbf{j}$ and $(\mathbf{i} - 2\mathbf{j}) + (\mathbf{i} + \mathbf{j}) = 2\mathbf{i} - \mathbf{j}$. By (1.6), (total mass is $M = 8$)

$$\mathbf{c}_M = \tfrac{1}{8}(2(\mathbf{0}) + 2(\mathbf{i} - 2\mathbf{j}) + 2(\mathbf{i} + \mathbf{j}) + 2(2\mathbf{i} - \mathbf{j})) = \tfrac{1}{8}(8\mathbf{i} - 4\mathbf{j}) = \mathbf{i} - \tfrac{1}{2}\mathbf{j}.$$

Therefore, the center of mass of the system is located at $(1, -1/2)$.

16. By (1.6), $\mathbf{c}_M = \frac{1}{M}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3)$, where $\mathbf{c}_M = (1, 1) = \mathbf{i} + \mathbf{j}$, $M = 2 + 2 + 3 = 7$, $m_1 = 2$, $\mathbf{r}_1 = -\mathbf{i} + 3\mathbf{j}$, $m_2 = 2$, $\mathbf{r}_2 = \mathbf{i} - 2\mathbf{j}$ and $m_3 = 3$. The vector $\mathbf{r}_3 = x\mathbf{i} + y\mathbf{j}$ needs to be determined. From

$$\mathbf{i} + \mathbf{j} = \tfrac{1}{7}\Big(2(-\mathbf{i} + 3\mathbf{j}) + 2(\mathbf{i} - 2\mathbf{j}) + 3(x\mathbf{i} + y\mathbf{j})\Big),$$

we get (multiply by 7 and consider $\mathbf{i}$ and $\mathbf{j}$ components), $7 = -2 + 2 + 3x$ and $7 = 6 - 4 + 3y$. Thus, $x = 7/3$ and $y = 5/3$. The third mass is placed at $(7/3, 5/3)$.

17. By (1.6), $\mathbf{c}_M = \frac{1}{M}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3)$, where $\mathbf{c}_M = (1, 0)$, $m_1 = 2$, $\mathbf{r}_1 = (-1, 0)$, $m_2 = 2$, $\mathbf{r}_2 = (1, -2)$ and $\mathbf{r}_3 = (2, 1)$. We need to find m_3. Note that $M = 2 + 2 + m_3 = 4 + m_3$. From

$$(1, 0) = \tfrac{1}{4 + m_3}\Big(2(-1, 0) + 2(1, -2) + m_3(2, 1)\Big),$$

we get

$$1 = \frac{2m_3}{4 + m_3} \qquad \text{and} \qquad 0 = \frac{-4 + m_3}{4 + m_3}.$$

Solving either of the two equations, we get $m_3 = 4$.

18. The direction from $(0, 0)$ to $(1, 1)$ is represented by the (unit) vector $(\mathbf{i} + \mathbf{j})/\sqrt{2}$; so $\mathbf{F}_1 = ||\mathbf{F}_1||(\mathbf{i} + \mathbf{j})/\sqrt{2} = 3(\mathbf{i} + \mathbf{j})/\sqrt{2}$. Similarly, the direction from $(0, 0)$ to $(0, -4)$ is represented by the (unit) vector $-\mathbf{j}$; hence $\mathbf{F}_2 = ||\mathbf{F}_2||(-\mathbf{j}) = -4\mathbf{j}$. Finally, the direction from $(0, 0)$ to $(10, 1)$ is represented by $(10\mathbf{i} + \mathbf{j})/\sqrt{101}$; so, since $||\mathbf{F}_3|| = 6$, we get $\mathbf{F}_3 = 6(10\mathbf{i} + \mathbf{j})/\sqrt{101}$. It follows that the resultant force of this system is $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = (3/\sqrt{2} + 60/\sqrt{101})\mathbf{i} + (3/\sqrt{2} - 4 + 6/\sqrt{101})\mathbf{j} \approx 8.09154\mathbf{i} - 1.28166\mathbf{j}$.

19. Place the triangle in the first quadrant (call it the triangle ABC) so that its vertex A is at the origin and the side $\overline{AB}$ lies on the x-axis — thus $A(0,0)$ and $B(2,0)$; see Figure 19. To find the coordinates of $C(c_1, c_2)$ use the fact that the distances from C to A and from C to B must be 2. Since $d(A,C) = 2$, it follows that $d(A,C)^2 = 4$ and $c_1^2 + c_2^2 = 4$. Similarly, $d(B,C) = 2$ implies that $(c_1 - 2)^2 + c_2^2 = 4$. Subtracting the second equation from the first we get $(c_1 - 2)^2 - c_1^2 = 0$ and $c_1 = 1$. Consequently, $c_2 = \sqrt{3}$ and the coordinates of C are $(1, \sqrt{3})$.

We decided to solve this question so as to practice the distance formula. There are other ways of getting C: for example, by symmetry, the x-coordinate of C must be half-way between 0 and 2; i.e., $c_1 = 1$. The y-coordinate is the height of an equilateral triangle of side 2; i.e., $c_2 = \sqrt{2^2 - 1^2} = \sqrt{3}$ by the Pythagorean Theorem. Alternatively, using trigonometry, $c_1 = 2\cos(\pi/3) = 2(1/2) = 1$ and $c_2 = 2\sin(\pi/3) = 2(\sqrt{3}/2) = 1\sqrt{3}$.

All masses are 1 kg. The position vectors are $\mathbf{r}_A = \mathbf{0}$, $\mathbf{r}_B = 2\mathbf{i}$ and $\mathbf{r}_C = \mathbf{i} + \sqrt{3}\mathbf{j}$. Since the total mass is $M = 3$, we get that

$$\mathbf{c}_M = \tfrac{1}{3}(1(\mathbf{0}) + 1(2\mathbf{i}) + 1(\mathbf{i} + \sqrt{3}\mathbf{j})) = \tfrac{1}{3}(3\mathbf{i} + \sqrt{3}\mathbf{j}) = \mathbf{i} + \tfrac{\sqrt{3}}{3}\mathbf{j}.$$

In words, the center of mass $\mathbf{c}_M$ is located on the axis of symmetry of the triangle (one such axis is the line through C perpendicular to $\overline{AB}$), $\sqrt{3}/3$ units away from the side $\overline{AB}$.

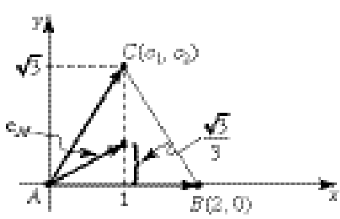

Figure 19

20. Let $\mathbf{v}_A$ be the velocity of the slower cat (walking, say, along the positive x-axis) and $\mathbf{v}_B$ the velocity of the faster cat (walking along the positive y-axis), and assume that both cats start from the origin. Then $\mathbf{v}_A = 10\mathbf{i}$ and $\mathbf{v}_B = 12\mathbf{j}$ and the relative velocity of the faster cat as seen by the slower cat is $\mathbf{v}_B - \mathbf{v}_A = -10\mathbf{i} + 12\mathbf{j}$.

21. The forces are

$$\mathbf{F}_1 = 5(\cos(\pi/10)\mathbf{i} + \sin(\pi/10)\mathbf{j}) \approx 4.755\mathbf{i} + 1.545\mathbf{j},$$
$$\mathbf{F}_2 = 5(\cos(\pi/5)\mathbf{i} + \sin(\pi/5)\mathbf{j}) \approx 4.045\mathbf{i} + 2.939\mathbf{j},$$
$$\mathbf{F}_3 = 5(\cos(\pi/2)\mathbf{i} + \sin(\pi/2)\mathbf{j}) = 5\mathbf{j}$$

and

$$\mathbf{F}_4 = 5(\cos(13\pi/10)\mathbf{i} + \sin(13\pi/10)\mathbf{j}) \approx -2.939\mathbf{i} - 4.045\mathbf{j}.$$

Their resultant is $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 \approx 5.861\mathbf{i} + 5.439\mathbf{j}$.

1.3. The Dot Product

1. Let $\mathbf{v} = (v_1, \dots, v_n)$ and $\mathbf{w} = (w_1, \dots, w_n)$. Then $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + \dots + v_n w_n$ and $\mathbf{w} \cdot \mathbf{v} = w_1 v_1 + \dots + w_n v_n$ are equal due to commutativity of multiplication of real numbers.

2. Since $\mathbf{v} \cdot \mathbf{w}$ is a real number, $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$ is a vector parallel to $\mathbf{u}$. On the other hand, $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$ is a scalar multiple of $\mathbf{w}$, hence parallel to it. So, in general, $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w}) \neq (\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$.

3. Let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$. Then $\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$ and

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = u_1(v_1 + w_1) + u_2(v_2 + w_2) + u_3(v_3 + w_3).$$

Since

$$\mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_1 w_1 + u_2 w_2 + u_3 w_3,$$

it follows (by distributivity of multiplication of real numbers) that $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.

The remaining two identities are proven in a similar way. From $\mathbf{u} = (u_1, u_2, u_3)$ we get $\alpha\mathbf{u} = (\alpha u_1, \alpha u_2, \alpha u_3)$ and

$$(\alpha\mathbf{u}) \cdot \mathbf{v} = (\alpha u_1)v_1 + (\alpha u_2)v_2 + (\alpha u_3)v_3 = \alpha u_1 v_1 + \alpha u_2 v_2 + \alpha u_3 v_3.$$

Similarly,

$$\alpha(\mathbf{u} \cdot \mathbf{v}) = \alpha(u_1 v_1 + u_2 v_2 + u_3 v_3)$$

and the identity $(\alpha\mathbf{u}) \cdot \mathbf{v} = \alpha(\mathbf{u} \cdot \mathbf{v})$ follows from the distributivity of multiplication of real numbers.

To prove the third identity we use the commutativity of the dot product (twice) and the identity that we have just proved:

$$\mathbf{u} \cdot (\alpha\mathbf{v}) = (\alpha\mathbf{v}) \cdot \mathbf{u} = \alpha(\mathbf{v} \cdot \mathbf{u}) = \alpha(\mathbf{u} \cdot \mathbf{v}).$$

4. Let $\mathbf{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$. Then

$$\mathbf{v} \cdot \mathbf{v} = (v_1, v_2, \dots, v_n) \cdot (v_1, v_2, \dots, v_n) = v_1^2 + v_2^2 + \dots + v_n^2 = \|\mathbf{v}\|^2.$$

5. From linear algebra we know that any vector in $\mathbb{R}^3$ can be written as a linear combination of three (non-zero) mutually orthogonal vectors; i.e., $\mathbf{a} = a_{\mathbf{u}}\mathbf{u} + a_{\mathbf{v}}\mathbf{v} + a_{\mathbf{w}}\mathbf{w}$, where $a_{\mathbf{u}}$, $a_{\mathbf{v}}$ and $a_{\mathbf{w}}$ are real numbers. Computing the dot product of $\mathbf{a}$ with $\mathbf{u}$, we get

$$\begin{aligned}\mathbf{a} \cdot \mathbf{u} &= (a_{\mathbf{u}}\mathbf{u} + a_{\mathbf{v}}\mathbf{v} + a_{\mathbf{w}}\mathbf{w}) \cdot \mathbf{u} \\ &= a_{\mathbf{u}}\mathbf{u} \cdot \mathbf{u} + a_{\mathbf{v}}\mathbf{v} \cdot \mathbf{u} + a_{\mathbf{w}}\mathbf{w} \cdot \mathbf{u} = a_{\mathbf{u}}\mathbf{u} \cdot \mathbf{u},\end{aligned}$$

since (by orthogonality) the remaining two terms on the right side vanish. Hence $\mathbf{a}\cdot\mathbf{u} = a_{\mathbf{u}}||\mathbf{u}||^2$ and $a_{\mathbf{u}} = \mathbf{a}\cdot\mathbf{u}/||\mathbf{u}||^2$. The expressions for $a_{\mathbf{v}}$ and $a_{\mathbf{w}}$ are obtained analogously, by computing the dot product of $\mathbf{a}$ with $\mathbf{v}$ and $\mathbf{w}$.

6. Let $\mathbf{v} = 2\mathbf{j} - \mathbf{k}$ and $\mathbf{w} = \mathbf{i} + \mathbf{j} - 3\mathbf{k}$. From

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\,||\mathbf{w}||} = \frac{(2\mathbf{j}-\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}-3\mathbf{k})}{\sqrt{5}\sqrt{11}} = \frac{5}{\sqrt{55}}$$

it follows that $\theta = \arccos(5/\sqrt{55}) \approx .0.8309$ rad.

7. Let θ_1 denote the angle at $(0,3,4)$: it is the angle between the vectors from $(0,3,4)$ to $(0,3,0)$ (call it $\mathbf{u}_1$) and from $(0,3,4)$ to $(12,0,5)$ (call it $\mathbf{v}_1$). It follows that $\mathbf{u}_1 = (0,0,-4)$ and $\mathbf{v}_1 = (12,-3,1)$ and

$$\cos\theta_1 = \frac{\mathbf{u}_1\cdot\mathbf{v}_1}{||\mathbf{u}_1||\,||\mathbf{v}_1||} = \frac{-4}{4\sqrt{154}} = \frac{-1}{\sqrt{154}}$$

and therefore $\theta_1 = \arccos(-1/\sqrt{154}) \approx 1.6515$ rad. Denote by θ_2 the angle at $(0,3,0)$; that is, θ_2 is the angle between the vectors $\mathbf{u}_2 = (0,0,4)$ (from $(0,3,0)$ to $(0,3,4)$) and $\mathbf{v}_2 = (12,-3,5)$ (from $(0,3,0)$ to $(12,0,5)$). We get

$$\cos\theta_2 = \frac{\mathbf{u}_2\cdot\mathbf{v}_2}{||\mathbf{u}_2||\,||\mathbf{v}_2||} = \frac{20}{4\sqrt{178}}$$

and $\theta_2 = \arccos(5/\sqrt{178}) \approx 1.1867$ rad. Since $\theta_1 + \theta_2 + \theta_3 = \pi$, it follows that $\theta_3 = 0.3034$ rad.

8. Let $\mathbf{v} = (3,2,4)$ and $\mathbf{w} = (1,2,7)$. Since $\mathbf{v}$ is not a scalar multiple of $\mathbf{w}$ (or the other way around), $\mathbf{v}$ and $\mathbf{w}$ are not parallel. Moreover, $\mathbf{v}\cdot\mathbf{w} = (3,2,4)\cdot(1,2,7) = 35 \neq 0$ implies that they are not orthogonal either. From $(\mathbf{i}-\mathbf{j}-\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}) = 0$ it follows that $\mathbf{i}-\mathbf{j}-\mathbf{k}$ and $\mathbf{i}+\mathbf{j}$ are orthogonal.

9. Computing the dot product of the given vectors, we get

$$\left(\mathbf{w} - \frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||^2}\mathbf{v}\right)\cdot\mathbf{v} = \mathbf{w}\cdot\mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||^2}\mathbf{v}\cdot\mathbf{v} = \mathbf{w}\cdot\mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||^2}||\mathbf{v}||^2 = 0,$$

by the distributivity and the commutativity of the dot product, and the fact that $\mathbf{v}\cdot\mathbf{v} = ||\mathbf{v}||^2$.

10. Let $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$ be vectors in $\mathbb{R}^2$. If $\mathbf{v} = \mathbf{0}$, or $\mathbf{w} = \mathbf{0}$, or both are zero, then both sides of the inequality are zero. So assume that $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{w} \neq \mathbf{0}$. From $\mathbf{v}\cdot\mathbf{w} = v_1w_1 + v_2w_2$ and $||\mathbf{v}||\,||\mathbf{w}|| = \sqrt{v_1^2+v_2^2}\sqrt{w_1^2+w_2^2}$ it follows that

$$(\mathbf{v}\cdot\mathbf{w})^2 = v_1^2w_1^2 + v_2^2w_2^2 + 2v_1w_1v_2w_2$$

and

$$||\mathbf{v}||^2||\mathbf{w}||^2 = v_1^2w_1^2 + v_2^2w_2^2 + v_1^2w_2^2 + v_2^2w_1^2.$$

So, in order to prove $|\mathbf{v}\cdot\mathbf{w}|^2 \le ||\mathbf{v}||^2||\mathbf{w}||^2$ (and hence the given inequality, by computing the square root) we have to show that $2v_1w_1v_2w_2 \le v_1^2w_2^2 + v_2^2w_1^2$. Expanding (the obvious) inequality $(v_1w_2 - v_2w_1)^2 \ge 0$ we get $v_1^2w_2^2 + v_2^2w_1^2 - 2v_1w_1v_2w_2 \ge 0$, and we are done.

The equality holds when $v_1w_2 - v_2w_1 = 0$, i.e., if and only if $v_1w_2 = v_2w_1$ or $v_1/w_1 = v_2/w_2$. If we set $v_1/w_1 = \alpha$, then $v_1 = \alpha w_1$; from $v_2/w_2 = \alpha$, then $v_2 = \alpha w_2$. In words, equality holds when $\mathbf{v}$ and $\mathbf{w}$ are parallel. (If $v_1 = 0$, then $v_2 \ne 0$ (by assumption, $\mathbf{v} \ne \mathbf{0}$ so at least one component must be non-zero) and hence $v_1w_2 - v_2w_1 = 0$ implies $w_1 = 0$; hence $\mathbf{v} = (0, v_2)$ and $\mathbf{w} = (0, w_2)$ are parallel. Other cases with one of the coordinates equal to zero are dealt with analogously.) The proof for vectors in $\mathbb{R}^3$ is very similar but messier.

Alternatively, we use the definition $\mathbf{v}\cdot\mathbf{w} = ||\mathbf{v}||\,||\mathbf{w}||\cos\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$. Then

$$|\mathbf{v}\cdot\mathbf{w}| = ||\mathbf{v}||\,||\mathbf{w}||\,|\cos\theta| \le ||\mathbf{v}||\,||\mathbf{w}||,$$

since $|\cos\theta| \le 1$. The equality holds when $\cos\theta = \pm 1$; i.e., when $\mathbf{v}$ and $\mathbf{w}$ are parallel.

11. Consider the cube spanned by $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ with one vertex at the origin (as far as angles are concerned, we are free to choose any cube). In that case, the (direction of the) diagonal can be expressed as $\mathbf{i}+\mathbf{j}+\mathbf{k}$, and hence the angle θ between the diagonal and the side represented by $\mathbf{i}$ is

$$\cos\theta = \frac{\mathbf{i}\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})}{||\mathbf{i}||\,||\mathbf{i}+\mathbf{j}+\mathbf{k}||} = \frac{1}{\sqrt{3}},$$

i.e., $\theta = \arccos(1/\sqrt{3}) \approx 0.9553$ rad. The remaining two angles (between the diagonal and the sides represented by $\mathbf{j}$ and $\mathbf{k}$) are computed in the same way, and are both equal to θ.

12. We use the following fact: if a parallelogram is spanned by vectors $\mathbf{v}$ and $\mathbf{w}$, then its diagonals are given by $\mathbf{v}+\mathbf{w}$ and $\mathbf{w}-\mathbf{v}$ (or $\mathbf{v}-\mathbf{w}$).

Let $\mathbf{v}$ be the vector from $(0,2,-3)$ to $(1,1,0)$ (hence $\mathbf{v} = (1,-1,3)$) and $\mathbf{w}$ be the vector from $(0,2,-3)$ to $(-1,0,1)$ (hence $\mathbf{w} = (-1,-2,4)$). In this case, the fourth vertex of the parallelogram is at $(0,-3,7)$ (which is the tip of $\mathbf{v}+\mathbf{w}$). The angle between the diagonals satisfies

$$\cos\theta = \frac{(\mathbf{v}+\mathbf{w})\cdot(\mathbf{w}-\mathbf{v})}{||\mathbf{v}+\mathbf{w}||\,||\mathbf{w}-\mathbf{v}||} = \frac{(0,-3,7)\cdot(-2,-1,1)}{\sqrt{58}\sqrt{6}} = \frac{10}{\sqrt{348}},$$

i.e., $\theta = \arccos(10/\sqrt{348}) \approx .1.0050$ rad.

We have to consider two more cases. Let $\overline{\mathbf{v}}$ be the vector from $(1,1,0)$ to $(0,2,-3)$ (hence $\overline{\mathbf{v}} = (-1,1,-3)$) and $\overline{\mathbf{w}}$ be the vector from $(1,1,0)$ to $(-1,0,1)$ (hence $\overline{\mathbf{w}} = (-2,-1,1)$). In this case, the fourth vertex of the parallelogram is at $(-3,0,-2)$. The angle between the diagonals satisfies

$$\cos\overline{\theta} = \frac{(\overline{\mathbf{v}}+\overline{\mathbf{w}})\cdot(\overline{\mathbf{w}}-\overline{\mathbf{v}})}{||\overline{\mathbf{v}}+\overline{\mathbf{w}}||\,||\overline{\mathbf{w}}-\overline{\mathbf{v}}||} = \frac{(-3,0,-2)\cdot(-1,-2,4)}{\sqrt{13}\sqrt{21}} = \frac{-5}{\sqrt{273}},$$

i.e., $\overline{\theta} = \arccos(-5/\sqrt{273}) \approx 1.8782$ rad. The needed angle is $\pi - 1.8782 \approx 1.2634$ rad.

Finally, let $\tilde{\mathbf{v}}$ be the vector from $(-1,0,1)$ to $(0,2,-3)$ (hence $\tilde{\mathbf{v}} = (1,2,-4)$) and $\tilde{\mathbf{w}}$ be the vector from $(-1,0,1)$ to $(1,1,0)$ (hence $\tilde{\mathbf{w}} = (2,1,-1)$). In this case, the fourth vertex of the parallelogram is at $(3,3,-5)$. The angle between the diagonals satisfies

$$\cos\tilde{\theta} = \frac{(\tilde{\mathbf{v}}+\tilde{\mathbf{w}})\cdot(\tilde{\mathbf{w}}-\tilde{\mathbf{v}})}{||\tilde{\mathbf{v}}+\tilde{\mathbf{w}}||\,||\tilde{\mathbf{w}}-\tilde{\mathbf{v}}||} = \frac{(3,3,-5)\cdot(1,-1,3)}{\sqrt{43}\sqrt{11}} = \frac{-15}{\sqrt{473}},$$

i.e., $\tilde{\theta} = \arccos(-15/\sqrt{473}) \approx 2.3319$ rad. The angle is $\pi - 2.3319 \approx 0.8097$ rad.

13. The angle $\theta_{\mathbf{v}}$ between $\mathbf{v}$ and $\mathbf{a}$ is

$$\begin{aligned}\cos\theta_{\mathbf{v}} &= \frac{\mathbf{v}\cdot\mathbf{a}}{||\mathbf{v}||\,||\mathbf{a}||} = \frac{\mathbf{v}\cdot(||\mathbf{v}||\mathbf{w}+||\mathbf{w}||\mathbf{v})}{||\mathbf{v}||\,||\mathbf{a}||} = \frac{||\mathbf{v}||\mathbf{v}\cdot\mathbf{w}+||\mathbf{w}||\mathbf{v}\cdot\mathbf{v}}{||\mathbf{v}||\,||\mathbf{a}||}\\ &= \frac{||\mathbf{v}||(\mathbf{v}\cdot\mathbf{w}+||\mathbf{v}||^2)}{||\mathbf{v}||\,||\mathbf{a}||} = \frac{\mathbf{v}\cdot\mathbf{w}+||\mathbf{v}||^2}{||\mathbf{a}||},\end{aligned}$$

since $||\mathbf{v}|| = ||\mathbf{w}||$. Similarly, the angle $\theta_{\mathbf{w}}$ between $\mathbf{w}$ and $\mathbf{a}$ is computed to be

$$\begin{aligned}\cos\theta_{\mathbf{w}} &= \frac{\mathbf{w}\cdot\mathbf{a}}{||\mathbf{w}||\,||\mathbf{a}||} = \frac{\mathbf{w}\cdot(||\mathbf{v}||\mathbf{w}+||\mathbf{w}||\mathbf{v})}{||\mathbf{w}||\,||\mathbf{a}||} = \frac{||\mathbf{v}||\mathbf{w}\cdot\mathbf{w}+||\mathbf{w}||\mathbf{w}\cdot\mathbf{v}}{||\mathbf{w}||\,||\mathbf{a}||}\\ &= \frac{||\mathbf{w}||(||\mathbf{w}||^2+\mathbf{w}\cdot\mathbf{v})}{||\mathbf{w}||||\mathbf{a}||} = \frac{||\mathbf{w}||^2+\mathbf{w}\cdot\mathbf{v}}{||\mathbf{a}||}.\end{aligned}$$

By assumption, $||\mathbf{v}|| = ||\mathbf{w}||$, and therefore $\cos\theta_{\mathbf{v}} = \cos\theta_{\mathbf{w}}$, and since (in the definition of the angle between two vectors there is a requirement that $0 \le \theta_{\mathbf{v}}, \theta_{\mathbf{w}} < \pi$) it follows that $\theta_{\mathbf{v}} = \theta_{\mathbf{w}}$.

14. The work of a force $\mathbf{F}$ acting at an angle θ on an object is given by $W = \mathbf{F}\cdot\mathbf{d}$, where $\mathbf{d}$ is the displacement vector. In our case, $\mathbf{d} = (3-1,-2-1) = (2,-3)$ (so $||\mathbf{d}|| = \sqrt{13}$), $||\mathbf{F}|| = 10$, $\theta = \pi/3$ and hence $W = 10\sqrt{13}\cos(\pi/3) = 5\sqrt{13}$.

15. Let $\mathbf{a} = \mathbf{i}+2\mathbf{j}$, $\mathbf{v} = \mathbf{i}+\mathbf{j}$ and $\mathbf{w} = \mathbf{i}-\mathbf{j}$. Vectors $\mathbf{v}$ and $\mathbf{w}$ are orthogonal ($\mathbf{v}\cdot\mathbf{w} = 0$) and hence $\mathbf{a} = a_{\mathbf{v}}\mathbf{v} + a_{\mathbf{w}}\mathbf{w}$, where $a_{\mathbf{v}}\mathbf{v} = pr_{\mathbf{v}}\mathbf{a} = (\mathbf{a}\cdot\mathbf{v}/||\mathbf{v}||^2)\mathbf{v}$ and $a_{\mathbf{w}}\mathbf{w} = pr_{\mathbf{w}}\mathbf{a} = (\mathbf{a}\cdot\mathbf{w}/||\mathbf{w}||^2)\mathbf{w}$ are the vector orthogonal projections of $\mathbf{a}$ onto $\mathbf{v}$ and $\mathbf{w}$; see (1.11) and the text following it. Since

$$a_{\mathbf{v}}\mathbf{v} = \frac{(\mathbf{i}+2\mathbf{j})\cdot(\mathbf{i}+\mathbf{j})}{||\mathbf{i}+\mathbf{j}||^2}(\mathbf{i}+\mathbf{j}) = \tfrac{3}{2}(\mathbf{i}+\mathbf{j})$$

and

$$a_{\mathbf{v}}\mathbf{w} = \frac{(\mathbf{i}+2\mathbf{j})\cdot(\mathbf{i}-\mathbf{j})}{||\mathbf{i}-\mathbf{j}||^2}(\mathbf{i}-\mathbf{j}) = -\tfrac{1}{2}(\mathbf{i}-\mathbf{j}),$$

the desired decomposition is

$$\mathbf{a} = \mathbf{i}+2\mathbf{j} = \tfrac{3}{2}(\mathbf{i}+\mathbf{j}) - \tfrac{1}{2}(\mathbf{i}-\mathbf{j}).$$

16. Let $\mathbf{u} = (\mathbf{i}+\mathbf{j}-\mathbf{k})/\sqrt{3}$, $\mathbf{v} = (\mathbf{i}-\mathbf{j})/\sqrt{2}$ and $\mathbf{w} = (\mathbf{i}+\mathbf{j}+2\mathbf{k})/\sqrt{6}$. First, we check that all vectors have length 1: $||\mathbf{u}|| = \sqrt{(1/\sqrt{3})^2+(1/\sqrt{3})^2+(-1/\sqrt{3})^2} = \sqrt{3/3} = 1$, $||\mathbf{v}|| =$

$\sqrt{(1/\sqrt{2})^2 + (-1/\sqrt{2})^2} = \sqrt{2/2} = 1$ and $||\mathbf{w}|| = \sqrt{(1/\sqrt{6})^2 + (1/\sqrt{6})^2 + (2/\sqrt{6})^2} = \sqrt{6/6} = 1$. From $\mathbf{u} \cdot \mathbf{v} = (1/\sqrt{3})(1/\sqrt{2})(\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot (\mathbf{i} - \mathbf{j}) = (1/\sqrt{3})(1/\sqrt{2})(1 - 1) = 0$, $\mathbf{u} \cdot \mathbf{w} = (1/\sqrt{3})(1/\sqrt{6})(\mathbf{i}+\mathbf{j}-\mathbf{k}) \cdot (\mathbf{i}+\mathbf{j}+2\mathbf{k}) = (1/\sqrt{3})(1/\sqrt{6})(1+1-2) = 0$ and $\mathbf{v} \cdot \mathbf{w} = (1/\sqrt{2})(1/\sqrt{6})(\mathbf{i} - \mathbf{j}) \cdot (\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = (1/\sqrt{2})(1/\sqrt{6})(1 - 1) = 0$ it follows that the three vectors are orthogonal to each other. Therefore, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is an orthonormal set of vectors.

17. Using the formula of Example 1.26, we get $3(x - 0) + 0(y + 2) - 1(z - 2) = 0$; i.e., $3x - z + 2 = 0$.

18. According to Example 1.26, we need to find a normal vector to the given plane. In other words, we need to find a vector $\mathbf{n} = (a, b, c)$ that is perpendicular to both $\mathbf{v}_1 = (1, 6, 0)$ (vector from $(0, -2, 3)$ to $(1, 4, 3)$) and $\mathbf{v}_2 = (-1, 9, -3)$ (vector from $(0, -2, 3)$ to $(-1, 7, 0)$). From $\mathbf{n} \cdot \mathbf{v}_1 = 0$ and $\mathbf{n} \cdot \mathbf{v}_2 = 0$ we get $a + 6b = 0$ and $-a + 9b - 3c = 0$. Pick $b = 1$; then $a = -6$, and from $6 + 9 - 3c = 0$ we get $c = 5$. Thus, $\mathbf{n} = (-6, 1, 5)$, and the equation (1.12) of the plane is

$$-6(x - 0) + 1(y + 2) + 5(z - 3) = 0,$$

i.e., $-6x + y + 5z - 13 = 0$.

Comments: we could have chosen vectors that lie in the plane in several ways; also, we could have used any of the given points to substitute into the equation (1.12). Of course, in any case, we get the same equation. In Section 1.5 we will discover an easier way of computing a normal vector, using the cross product of vectors.

19. The fact that the plane must be perpendicular to the line $\mathbf{l}(t) = (2 + 3t, 1 - t, 7)$, $t \in \mathbb{R}$, implies that the vector $(3, -1, 0)$ (direction vector of the line) is, at the same time, normal vector to the plane. Using (1.12), we get $3(x+2) - 1(y-1) + 0(z-4) = 0$, i.e., $3x - y + 7 = 0$.

20. From the definition, we get $\mathbf{v} \cdot \mathbf{u} = ||\mathbf{v}|| \, ||\mathbf{u}|| \cos\theta = ||\mathbf{v}|| \cos\theta$, since $||\mathbf{u}|| = 1$ by assumption. From the fact that $\cos\theta \le 1$ we conclude that $\mathbf{v} \cdot \mathbf{u}$ is the largest when $\cos\theta = 1$; i.e., when $\theta = 0$. In other words, $\mathbf{v} \cdot \mathbf{u}$ is the largest when $\mathbf{u}$ is parallel to $\mathbf{v}$ and of the same direction as $\mathbf{v}$; shortly, when $\mathbf{u} = \mathbf{v}/||\mathbf{v}||$. Similarly, $\mathbf{v} \cdot \mathbf{u}$ is the smallest when $\cos\theta = -1$; i.e., when $\theta = \pi$. In this case $\mathbf{u} = -\mathbf{v}/||\mathbf{v}||$ (so $\mathbf{u}$ is parallel to $\mathbf{v}$ but of the opposite direction). Since $\mathbf{u}$ and $\mathbf{v}$ are non-zero vectors, $\mathbf{v} \cdot \mathbf{u} = 0$ implies that $\theta = \pi/2$. So directions $\mathbf{u}$ such that $\mathbf{v} \cdot \mathbf{u} = 0$ are directions perpendicular to $\mathbf{v}$ (i.e., belong to the plane perpendicular to $\mathbf{v}$).

21. According to the formula of Example 1.26, we need a point (and we have it — the origin) and a normal vector $\mathbf{n}$. Take $\mathbf{n}$ to be the vector from $(3, 2, -1)$ to $(0, 0, 7)$ (we could have taken the vector in the opposite direction — it would not make any difference). Then $\mathbf{n} = (-3, -2, 8)$ and the equation of the plane is $-3(x - 0) - 2(y - 0) + 8(z - 0) = 0$; i.e., $3x + 2y - 8z = 0$.

22. Let $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j}$. We have to compute the scalar projection of $\mathbf{a}$ onto the line passing through $(2, -1)$ and $(0, 4)$. The direction of that line is given by the vector $\mathbf{v} = -2\mathbf{i} + 5\mathbf{j}$, and so the scalar projection is

$$||pr_{\mathbf{v}}\mathbf{a}|| = \frac{|\mathbf{a} \cdot \mathbf{v}|}{||\mathbf{v}||} = \frac{|(3\mathbf{i} - 2\mathbf{j}) \cdot (-2\mathbf{i} + 5\mathbf{j})|}{||-2\mathbf{i} + 5\mathbf{j}||} = \frac{|-16|}{\sqrt{29}} \approx 2.9711.$$

23. Consider a constant force $\mathbf{F}$ moving an object around a triangle ABC: first from A to B, then to C and then back to A. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be the vectors represented by the directed line segments $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$ respectively. The work along $\overrightarrow{AB}$ is the dot product of the force and the displacement vector; i.e., it is equal to $\mathbf{F} \cdot \mathbf{a}$. Similarly, the work along the remaining two sides is $\mathbf{F} \cdot \mathbf{b}$ and $\mathbf{F} \cdot \mathbf{c}$, and the total work of $\mathbf{F}$ is $W = \mathbf{F} \cdot \mathbf{a} + \mathbf{F} \cdot \mathbf{b} + \mathbf{F} \cdot \mathbf{c} = \mathbf{F} \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c})$. Since $\mathbf{c} = -\mathbf{b} - \mathbf{a}$, it follows that $W = 0$.

An analogous proof works for any polygon. The key fact that is needed is that if the sides of a polygon are oriented so that the terminal point of one side is the initial point of the neighboring one (i.e., all sides are oriented counterclockwise, or all sides are oriented clockwise), then the vector sum of all directed line segments thus obtained is zero. As above, the work of $\mathbf{F}$ is the dot product of $\mathbf{F}$ and the vector sum of all sides, which is zero.

24. A vector $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$ that is parallel to the yz-plane must satisfy $u_1 = 0$; i.e., $\mathbf{u} = (0, u_2, u_3)$. Let $\mathbf{v} = \mathbf{i} + \mathbf{j} - 2\mathbf{k} = (1, 1, -2)$. From $\mathbf{u} \cdot \mathbf{v} = 0$ it follows that $u_2 - 2u_3 = 0$. Finally, $||\mathbf{u}|| = 1$ implies $||\mathbf{u}||^2 = 1$, and hence $u_2^2 + u_3^2 = 1$. Combining the two equations, we get $(2u_3)^2 + u_3^2 = 1$; i.e., $u_3^2 = 1/5$ and $u_3 = \pm 1/\sqrt{5}$. From $u_2 = 2u_3$ it follows that $u_2 = \pm 2/\sqrt{5}$. So there are two vectors that satisfy the given conditions: $2\mathbf{i}/\sqrt{5} + \mathbf{j}/\sqrt{5}$ and $-2\mathbf{i}/\sqrt{5} - \mathbf{j}/\sqrt{5}$.

25. From $\mathbf{u} \cdot \mathbf{v} = (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \cdot (2\mathbf{j} + \mathbf{k}) = 2 - 2 = 0$, $\mathbf{u} \cdot \mathbf{w} = (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \cdot (5\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = 5 - 1 - 4 = 0$ and $\mathbf{v} \cdot \mathbf{w} = (2\mathbf{j} + \mathbf{k}) \cdot (5\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = -2 + 2 = 0$, it follows that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are mutually orthogonal. From the three-dimensional version of Theorem 1.6 (the formula appears immediately after the proof; see also Exercise 5) we get (replace $\mathbf{a}$ by $\mathbf{i}$) $\mathbf{i} = i_{\mathbf{u}}\mathbf{u} + i_{\mathbf{v}}\mathbf{v} + i_{\mathbf{w}}\mathbf{w}$, where $i_{\mathbf{u}} = \mathbf{i} \cdot \mathbf{u}/||\mathbf{u}||^2 = 1/6$, $i_{\mathbf{v}} = \mathbf{i} \cdot \mathbf{v}/||\mathbf{v}||^2 = 0$ and $i_{\mathbf{w}} = \mathbf{i} \cdot \mathbf{w}/||\mathbf{w}||^2 = 1/6$. Thus $\mathbf{i} = \mathbf{u}/6 + \mathbf{w}/6$.

26. We want to find all unit vectors $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$ in $\mathbb{R}^3$ such that $\mathbf{u} \cdot \mathbf{w} = 0$ and $\mathbf{v} \cdot \mathbf{w} = 0$. From $\mathbf{u} \cdot \mathbf{w} = 0$ we get $w_1 - 2w_2 + w_3 = 0$, and from $\mathbf{v} \cdot \mathbf{w} = 0$ it follows that $w_1 + w_2 + w_3 = 0$. Subtracting the two equations we get that $-3w_2 = 0$, hence $w_2 = 0$. In that case, $w_3 = -w_1$ and so $w_1\mathbf{i} - w_1\mathbf{k}$ represents all vectors perpendicular to $\mathbf{u}$ and $\mathbf{v}$. Adjusting its length, we get

$$\mathbf{w} = \frac{w_1\mathbf{i} - w_1\mathbf{k}}{\sqrt{2w_1^2}} = \frac{1}{\sqrt{2}}\frac{w_1}{|w_1|}(\mathbf{i} - \mathbf{k}) = \pm\frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{k}).$$

27. We will show that any vector $\mathbf{a} \in \mathbb{R}^3$ perpendicular to all three of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ must be a zero vector. Since $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are mutually orthogonal, it follows that (see Theorem 1.6 and the text immediately following it) $\mathbf{a} = a_{\mathbf{u}}\mathbf{u} + a_{\mathbf{v}}\mathbf{v} + a_{\mathbf{w}}\mathbf{w}$, for some real numbers $a_{\mathbf{u}}$, $a_{\mathbf{v}}$ and $a_{\mathbf{w}}$. Computing the dot product of $\mathbf{a}$ with $\mathbf{u}$, we get $\mathbf{a}\cdot\mathbf{u} = a_{\mathbf{u}}\mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{u} + \mathbf{w}\cdot\mathbf{u}$. Since $\mathbf{a}$ must be orthogonal to $\mathbf{u}$ and $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are mutually orthogonal, it follows that $\mathbf{a}\cdot\mathbf{u} = 0$, $\mathbf{v}\cdot\mathbf{u} = 0$ and $\mathbf{w}\cdot\mathbf{u} = 0$, and so $0 = a_{\mathbf{u}}\|\mathbf{u}\|^2$ and $a_{\mathbf{u}} = 0$, since $\mathbf{u}$ is a non-zero vector by assumption. Computing the dot product of $\mathbf{a}$ with $\mathbf{v}$ and $\mathbf{w}$ and proceeding as above, we will get $a_{\mathbf{v}} = 0$ and $a_{\mathbf{w}} = 0$, so that $\mathbf{a} = \mathbf{0}$. Therefore, in $\mathbb{R}^3$, there cannot exist four (or more) mutually orthogonal vectors).

28. (a) Substituting given quantities into $\mathbf{p} = \mathbf{a} + t\mathbf{v} + s\mathbf{w}$, we get

$$(x, y, z) = (a_x, a_y, a_z) + t(v_x, v_y, v_z) + s(w_x, w_y, w_z).$$

Rewriting this equation in components, we obtain $x = a_x + tv_x + sw_x$, $y = a_y + tv_y + sw_y$ and $z = a_z + tv_z + sw_z$.

(b) From the equations for x and y in (a), we get $tv_x + sw_x = x - a_x$ and $tv_y + sw_y = y - a_y$. Multiplying the first equation by v_y, the second by $-v_x$, and adding them up, we get

$$s(v_y w_x - v_x w_y) = xv_y - yv_x - a_x v_y + a_y v_x,$$

and

$$s = \frac{xv_y - yv_x - a_x v_y + a_y v_x}{v_y w_x - v_x w_y}.$$

Similarly, we get

$$t = \frac{xw_y - yw_x - a_x w_y + a_y w_x}{v_x w_y - v_y w_x}.$$

Substituting expressions for s and t into $z = a_z + tv_z + sw_z$ and multiplying by the common denominator $v_y w_x - v_x w_y$, yields

$$(v_y w_x - v_x w_y)z = a_z(v_y w_x - v_x w_y) - (xw_y - yw_x)v_z + a_x v_z w_y - a_y v_z w_x$$
$$+ (xv_y - yv_x)w_z - a_x v_y w_z + a_y v_x w_z.$$

This equation can be simplified to $Ax + By + Cz + D = 0$, where $A = v_y w_z - v_z w_y$, $B = v_z w_x - v_x w_z$, $C = v_x w_y - v_y w_x$ and

$$D = a_x(v_z w_y - v_y w_z) - a_y(v_z w_x - v_x w_z) + a_z(v_y w_x - v_x w_y).$$

Note: familiarity with concepts that will be defined in Section 1.5 will help us recognize A, B and C as $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ components of the cross product of vectors $\mathbf{v}$ and $\mathbf{w}$. Since the cross product of $\mathbf{v}$ and $\mathbf{w}$ is perpendicular to both vectors, the vector (A, B, C) is normal to the plane. The coefficient $-D$ is equal to the determinant $\begin{vmatrix} a_x & a_y & a_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$, see Section 1.4.

29. Let $\mathbf{p} = (x, y, z)$. Then from $(x, y, z) = (2, 0, -1) + t(0, -1, 1) + s(3, 0, 1)$ we get $x = 2 + 3s$, $y = -t$ and $z = -1 + t + s$. Substituting $s = (x - 2)/3$ and $t = -y$ into the equation for z, we get $z = -1 - y + (x - 2)/3$, i.e., $-x + 3y + 3z + 5 = 0$.

30. We have to transform the equation $3x + y - z + 1 = 0$ into the equation of the form $\mathbf{p} = \mathbf{a} + t\mathbf{v} + s\mathbf{w}$, where $\mathbf{a}$ is a point in the plane and $\mathbf{v}$ and $\mathbf{w}$ are non-parallel vectors, perpendicular to the normal vector $\mathbf{n} = (3, 1, -1)$.

Take $x = 1$ and $y = 0$. From $3x + y - z + 1 = 0$ we get $z = 4$, and thus $\mathbf{a} = (1, 0, 4)$ is a point in the given plane.

We need to find a vector $\mathbf{v} = (v_x, v_y, v_z)$ from $\mathbf{n} \cdot \mathbf{v} = 3v_x + v_y - v_z = 0$. Take $v_x = 0$; then $v_y = v_z$. Thus, any vector of the form $(0, v_y, v_y)$, $v_y \neq 0$, is perpendicular to the normal vector $\mathbf{n} = (3, 1, -1)$. Take, for instance, $\mathbf{v} = (0, 2, 2)$. Next, we look for a vector $\mathbf{w} = (w_x, w_y, w_z)$ that satisfies $\mathbf{n} \cdot \mathbf{w} = 3w_x + w_y - w_z = 0$, and is not parallel to $\mathbf{v}$. Take $w_y = 0$ (that will guarantee that $\mathbf{w}$ is not parallel to $\mathbf{v}$!), then $w_z = 3w_x$. So, any vector of the form $(w_x, 0, 3w_x)$, $w_x \neq 0$, will do. Let $\mathbf{w} = (1, 0, 3)$. Thus, the desired equation is

$$\mathbf{p} = (1, 0, 4) + t(0, 2, 2) + s(1, 0, 3),$$

where $s, t \in \mathbb{R}$. Note: there is a faster way of obtaining parametric equations: recall that we need to solve $3x + y - z + 1 = 0$. We have one equation with three unknowns, so we take two as parameters, say, $x = s$ and $y = t$. Then $3s + t - z + 1 = 0$, i.e., $z = 3s + t + 1$. Thus,

$$(x, y, z) = (s, t, 3s + t + 1) = (0, 0, 1) + t(0, 1, 1) + s(1, 0, 3),$$

where $s, t \in \mathbb{R}$.

31. We need to solve $Ax + By + Cz + D = 0$ for x, y and z (clearly, at least one of A or B or C must be non-zero). Assuming that $C \neq 0$, we take $x = t$ and $y = s$. Then $At + Bs + Cz + D = 0$ implies that $z = -D/C - At/C - Bs/C$, and

$$(x, y, z) = (t, s, -\tfrac{D}{C} - \tfrac{A}{C}t - \tfrac{B}{C}s) = (0, 0, -\tfrac{D}{C}) + t(1, 0, -\tfrac{A}{C}) + s(0, 1, -\tfrac{B}{C}).$$

Thus, $\mathbf{p} = \mathbf{a} + t\mathbf{v} + s\mathbf{w}$, where $\mathbf{a} = (0, 0, -D/C)$, $\mathbf{v} = (1, 0, -A/C)$, $\mathbf{w} = (0, 1, -B/C)$ and $t, s \in \mathbb{R}$ (clearly, $\mathbf{v}$ and $\mathbf{w}$ are non-parallel). Similar parametric equations are obtained when $A \neq 0$ or $B \neq 0$.

1.4. Matrices and Determinants

1. Using elementary matrix operations,

$$2B - 16I_2 = 2\begin{bmatrix} 0 & 5 \\ 4 & 0 \end{bmatrix} - 16\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 10 \\ 8 & 0 \end{bmatrix} - \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix} = \begin{bmatrix} -16 & 10 \\ 8 & -16 \end{bmatrix}.$$

2. Using elementary matrix operations,

$$B^2 - 16I_2 + AC = \begin{bmatrix} 0 & 5 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 0 & 5 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 1 \\ 0 & -5 & 4 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 1 & 2 \\ 3 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix} - \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ 7 & -14 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 7 & -10 \end{bmatrix}.$$

3. Since C is a 3×2 matrix and B a 2×2 matrix, the product CB is defined and is a 3×2 matrix. The product BA is defined, and is a 2×3 matrix. Since CB and BA are not of the same type, the difference $CB - BA$ is not defined.

4. The product CA is a 3×3 matrix, and so the expression $I_3 - 3CA$ is defined and is a 3×3 matrix. We get

$$I_3 - 3CA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 3 \begin{bmatrix} -1 & 0 \\ 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 0 & -5 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 3 \begin{bmatrix} -2 & 1 & -1 \\ 2 & -11 & 9 \\ 6 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & 3 \\ -6 & 34 & -27 \\ -18 & -6 & 4 \end{bmatrix}.$$

5. The product AC is a 2×2 matrix, and consequently, the expression $AC + I_2$ is defined. Its value is

$$AC + I_2 = \begin{bmatrix} 2 & -1 & 1 \\ 0 & -5 & 4 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 2 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 7 & -14 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 7 & -13 \end{bmatrix}.$$

6. The product CA is a 3×3 matrix, and so is the sum $I_3 + CA$. Since A is a 2×3 matrix, the product $A(I_3 + CA)$ is defined and is a 2×3 matrix. We get

$$A(I_3 + CA) = \begin{bmatrix} 2 & -1 & 1 \\ 0 & -5 & 4 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 0 & -5 & 4 \end{bmatrix} \right)$$
$$= \begin{bmatrix} 2 & -1 & 1 \\ 0 & -5 & 4 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 1 & -1 \\ 2 & -11 & 9 \\ 6 & 2 & -1 \end{bmatrix} \right)$$
$$= \begin{bmatrix} 2 & -1 & 1 \\ 0 & -5 & 4 \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 2 & -10 & 9 \\ 6 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 14 & -11 \\ 14 & 58 & -45 \end{bmatrix}.$$

7. The product BA is defined, and is equal to the 2×3 matrix

$$BA = \begin{bmatrix} 0 & 5 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 0 & -5 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -25 & 20 \\ 8 & -4 & 4 \end{bmatrix}.$$

Since C is of type 3×2, the product $C(BA)$ is defined. Its value is

$$C(BA) = \begin{bmatrix} -1 & 0 \\ 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 0 & -25 & 20 \\ 8 & -4 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 25 & -20 \\ 16 & -33 & 28 \\ -8 & -71 & 56 \end{bmatrix}.$$

8. The product CA is defined, and is a 3×3 matrix. Since the number of columns of CA is not equal to the number of rows of B, the product $(CA)B$ is not defined.

9. The product AC is of type 2×2, hence $(AC)^2 = (AC)(AC)$ is also of type 2×2. Since B^2 is a 2×2 matrix, the expression $(AC)^2 + 4B^2$ is defined. From

$$AC = \begin{bmatrix} 2 & -1 & 1 \\ 0 & -5 & 4 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 7 & -14 \end{bmatrix}$$

we get

$$(AC)^2 = \begin{bmatrix} 0 & -3 \\ 7 & -14 \end{bmatrix} \begin{bmatrix} 0 & -3 \\ 7 & -14 \end{bmatrix} = \begin{bmatrix} -21 & 42 \\ -98 & 175 \end{bmatrix}.$$

Since

$$B^2 = \begin{bmatrix} 0 & 5 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 5 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix},$$

it follows that

$$(AC)^2 + 4B^2 = \begin{bmatrix} -21 & 42 \\ -98 & 175 \end{bmatrix} + \begin{bmatrix} 80 & 0 \\ 0 & 80 \end{bmatrix} = \begin{bmatrix} 59 & 42 \\ -98 & 255 \end{bmatrix}.$$

10. The assumption $A = B$ states that $a_{ij} = b_{ij}$ for all i, j. Adding c_{ij} to both sides, we get $a_{ij} + c_{ij} = b_{ij} + c_{ij}$ for all i, j. Since the corresponding entries in $A + C$ and $B + C$ are equal, it follows that $A + C = B + C$. Similarly, multiplying $a_{ij} = b_{ij}$ by α we get $\alpha a_{ij} = \alpha b_{ij}$ for all i, j, and so $\alpha A = \alpha B$.

11. We use properties of matrix operations. From $3A - X = I_2 + 4(C - X)$ it follows that $3X = I_2 + 4C - 3A$ and $X = (I_2 + 4C - 3A)/3$. Hence

$$X = \frac{1}{3}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} 6 & -3 \\ 12 & 0 \end{bmatrix} \right) = \frac{1}{3} \begin{bmatrix} -5 & 7 \\ -8 & 1 \end{bmatrix} = \begin{bmatrix} -5/3 & 7/3 \\ -8/3 & 1/3 \end{bmatrix}.$$

12. From $X = 4BC - X$ it follows that $2X = 4BC$ and so

$$X = 2BC = 2 \begin{bmatrix} 10 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2 \begin{bmatrix} 1 & 10 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 20 \\ 0 & 0 \end{bmatrix}.$$

13. Let $X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$. Then $AX = B$ implies

$$\begin{bmatrix} 2 & -1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 10 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 2x_{11} - x_{21} & 2x_{12} - x_{22} \\ 4x_{11} & 4x_{12} \end{bmatrix} = \begin{bmatrix} 10 & 1 \\ 0 & 0 \end{bmatrix}.$$

Comparing corresponding entries, we get $2x_{11} - x_{21} = 10$, $2x_{12} - x_{22} = 1$, $4x_{11} = 0$ and $4x_{12} = 0$. It follows that $x_{11} = x_{12} = 0$, and so $x_{21} = -10$ and $x_{22} = -1$, and hence $X = \begin{bmatrix} 0 & 0 \\ -10 & -1 \end{bmatrix}$.

14. By definition,

$$\mathbf{F}_A(4\mathbf{i}-3\mathbf{j}) = A\cdot\begin{bmatrix}4\\-3\end{bmatrix} = \begin{bmatrix}2 & -1\\4 & 0\end{bmatrix}\begin{bmatrix}4\\-3\end{bmatrix} = \begin{bmatrix}11\\16\end{bmatrix}.$$

Similarly,

$$\mathbf{F}_B(4\mathbf{i}-3\mathbf{j}) = B\cdot\begin{bmatrix}4\\-3\end{bmatrix} = \begin{bmatrix}10 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}4\\-3\end{bmatrix} = \begin{bmatrix}37\\0\end{bmatrix}.$$

15. The map $\mathbf{F}_C$ is given by $\mathbf{F}_C(\mathbf{v}) = C\cdot\mathbf{v}$, where $\mathbf{v}\in\mathbb{R}^2$. If $\mathbf{v}=(v_1,v_2)$, then

$$\mathbf{F}_C(\mathbf{v}) = \mathbf{F}_C\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right) = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}v_2\\v_1\end{bmatrix}.$$

In words, $\mathbf{F}_C$ switches the x- and y-components of a vector. Geometrically, $\mathbf{F}_C$ is a mapping that assigns to a vector its symmetric image with respect to the line $y=x$. Similarly,

$$\mathbf{F}_{I_2}(\mathbf{v}) = \mathbf{F}_{I_2}\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right) = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}v_1\\v_2\end{bmatrix};$$

i.e., $\mathbf{F}_{I_2}$ is the identity mapping: it maps a vector to itself.

16. Clearly,

$$A^2 = AA = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

and

$$AB = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}4 & 7\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}.$$

Consider $A = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$ and $B = \begin{bmatrix}0 & 0\\3 & 11\end{bmatrix}$. Their product is computed to be

$$AB = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\3 & 11\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}.$$

Remark: by replacing the entry 1 in A by a non-zero real number, and the entries 3 and 11 by two real numbers (at least one should be non-zero), we get an infinite number of non-zero matrices A and B such that $AB=0$.

17. By definition,

$$\det\begin{bmatrix}3 & 4\\4 & 3\end{bmatrix} = 3\cdot 3 - 4\cdot 4 = -7.$$

18. By definition,

$$\det\begin{bmatrix}-1 & 9\\0 & 1\end{bmatrix} = -1\cdot 1 - 9\cdot 0 = -1.$$

19. By definition,

$$\det\begin{bmatrix}\cos\theta & \sin\theta\\-r\sin\theta & r\cos\theta\end{bmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$