Chapter 1

Solutions

(1) The answers are all straightforward:

$$z_1^2 = (1+i)^2 = 1 + 2i - 1 = 2i$$

$$z_1^* = 1 - i$$

$$z_1 z_2^* = (1+i)(3+4i) = 3 + 3i + 4i - 4 = -1 + 7i$$

$$|z_1| = \sqrt{1+1} = \sqrt{2} \qquad \text{and} \qquad |z_2| = \sqrt{9+16} = 5$$

$$\begin{aligned} |z_1 z_2| &= |(1+i)(3-4i)| = |3 + 3i - 4i + 4| = |7 - i| = \sqrt{49+1} = 5\sqrt{2} \\ &= |z_1||z_2| \end{aligned}$$

(2) $[PV] = [\text{Force}]L^{-2}L^3 = MLT^{-2}]L^{-2}L^3 = ML^2T^{-2} = [\text{Energy}]$. From the ideal gas law, since N is dimensionless, kT must have the dimensions of energy. The SI unit of energy is Joule, so the units of k are Joule/K.

(3) First find $[k] = [\text{Force}]/[\text{Distance}] = MLT^{-2}L^{-1} = MT^{-2}$. (a) Write $\epsilon = m^x k^y A^z$, so

$$ML^2T^{-2} = M^x M^y T^{-2y} L^z = M^{x+y} T^{-2y} L^z$$

and $y = 1$, $z = 2$, and $x = 0$. That is, the energy scale is $\epsilon = kA^2$. You'll recall that the potential energy of a harmonic oscillator with amplitude A is $kA^2/2$ so the scale is the correct energy to within a factor of two. (b) Write $\epsilon = m^x k^y \hbar^z$, so

$$ML^2T^{-2} = M^x M^y T^{-2y} L^{2z} M^z T^{-z} = M^{x+y+z} T^{-2y-z} L^{2z}$$

and $z = 1$, $2y + z = 2y + 1 = 2$ so $y = 1/2$, and $x + y + z = x + 3/2 = 1$ so $x = -1/2$. Therefore $\epsilon = \hbar(k/m)^{1/2}$. In quantum mechanics you will learn that the energy spacing in the harmonic oscillator is indeed $\hbar\omega$ where $\omega = (k/m)^{1/2}$.

(4) The calculation is straightforward:

$$\tau = m^x \ell^y g^z$$

$$[\tau] = [m]^x[\ell]^y[g]^z$$
$$T = M^x L^y L^z T^{-2z} = M^x L^{y+z} T^{-2z}$$

Therefore $x = 0$, $y + z = 0$, $z = -1/2$, and so $y = -z = +1/2$. In other words

$$\tau = \ell^{1/2} g^{-1/2} = \sqrt{\frac{\ell}{g}}$$

This agrees with what you know from your introductory physics course, namely the period of a pendulum $T = 2\pi\sqrt{\ell/g}$. Dimensional analysis can't, of course, predict the factor of 2π.

(5) Newton's gravitational law gives $[G] = L^3 M^{-1} T^{-2}$, so write $\ell_P = G^x \hbar^y c^z$ which gives

$$L = L^{3x} M^{-x} T^{-2x} L^{2y} M^y T^{-y} L^z T^{-z} = L^{3x+2y+z} M^{-x+y} T^{-2x-y-z}$$

Therefore $x = y$, $2x + y + z = 3x + z = 0$, and $3x + 2y + z = 5x + z = 1$. Subtract the last two equations to get $x = 1/2 = y$, so $z = -3x = -3/2$ and $\ell_P = (G\hbar/c^3)^{1/2}$. This works out to be 1.6×10^{-35} m, 20 orders of magnitude smaller than the proton.

(6) The only parameter we have is H_0 which has units of inverse time, so the age of the universe would be $1/H_0 = (1/70)\, 3.1 \times 10^{19} = 4.4 \times 10^{17}$ sec = 14 billion years.

(7) Energy has dimensions ML^2T^{-2} and $[\hbar] = ML^2T^{-1}$ so write $E = m^x \hbar^y a^z$ giving

$$ML^2T^{-2} = M^x M^y L^{2y} T^{-y} L^z = M^{x+y} L^{2y+z} T^{-y}$$

so $y = 2$, $z = 2 - 2y = -2$, $x = 1 - y = -1$ and $E = \hbar^2/ma^2 = \hbar^2 c^2/(mc^2)a^2$. Therefore $E = (200\text{ MeV} \cdot \text{fm})^2/(0.511\text{ MeV}(10^5\text{ fm})^2) = 7.8 \times 10^{-6}$ MeV=7.8 eV, which is about right.

(8) Write $P = G^x M_\odot^y R^z$. Now $[P] = MLT^{-2}/L^2 = ML^{-1}T^{-2}$, so

$$ML^{-1}T^{-2} = M^{-x} L^{3x} T^{-2x} M^y L^z = M^{y-x} L^{3x+z} T^{-2x}$$

and it follows that $x = 1$. Also $y - x = 1$ so $y = 2$, and $3x + z = -1$ so $z = -4$ and we find

$$P = G\frac{M_\odot^2}{R^4} = 6.67 \times 10^{-11} \frac{(1.99 \times 10^{30})^2}{(6.96 \times 10^8)^4} \approx 10^{15}\text{ N/m}^2$$

Now $PV = nkT = (M_\odot/m_p)kT$ so the temperature at the center is

$$T = P\left(\frac{4}{3}\pi R^3\right)\frac{m_p}{M_\odot}\frac{1}{k} = 10^{15}\frac{4}{3}\pi(6.96 \times 10^8)^3 \frac{1.67 \times 10^{-27}}{1.99 \times 10^{30}} \frac{1}{1.38 \times 10^{-23}} \approx 10^8\text{ K}$$

(9) a) For $n = 0$, $f(x) = 1$ which does not change with x, so $f'(x) = 0$. (b) With $m = -n$,

$$f'(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{1}{(x+\Delta x)^m} - \frac{1}{x^m}\right] = -\lim_{\Delta x \to 0} \frac{m x^{m-1}\Delta x + \cdots}{x^m (x+\Delta x)^m} = -m x^{-m-1} = n x^{n-1}$$

(c) Put $y = f(x)$ so that $y^q = x^p$. Then $qy^{q-1}dy = px^{p-1}dx$ and

$$f'(x) = \frac{dy}{dx} = \frac{p}{q}x^{p-1}y^{1-q} = \frac{p}{q}x^{p-1}\left(x^{p/q}\right)^{1-q} = \frac{p}{q}x^{p-1}x^{p/q-p} = \frac{p}{q}x^{p/q-1}$$

(d) Any real number is infinitesimally close to a rational number on either side, so, sure, it makes sense that this rule holds for any real number exponent.

(10) We don't know the derivative of the $\cos^{-1}$ function, but we do know the derivative of cos, so write $\cos(y) = x$ and therefore $-\sin(y)\,dy = dx$ which gives

$$\frac{dy}{dx} = -\frac{1}{\sin(y)} = -\frac{1}{\sqrt{1-\cos^2(y)}} = -\frac{1}{\sqrt{1-x^2}}$$

(11) Differentiating gives $2ax + 2by(dy/dx) = 0$ so the points must satisfy $y = -ax/b$. Substituting into the ellipse equation gives $ax^2 + a^2x^2/b = (a + a^2/b)x^2 = c$. This gives the two solutions for $x = \pm\sqrt{c/(a + a^2/b)}$ with the corresponding values for y.

(12) Let x be the width of the pen, and y be the length, with the long side parallel to the wall. You have a constraint that $2x + y = 100$. The area of the pen is $A = xy$. So, to maximize A, calculate

$$\frac{dA}{dx} = \frac{d}{dx}x(100 - 2x) = \frac{d}{dx}(100x - 2x^2) = 100 - 4x = 0$$

Therefore $x = 25$ m, implying $y = 50$ m, maximizes the area of the pen.

(13) The volume is $V = xy^2$ and the area is $A = 3xy + 2y^2$. We could solve for x in terms of y and (the constant) A, and then find the derivative of V with respect to y, but I think it is easier to use implicit differentiation, as we did in Problem (4) above. That is

$$\frac{dV}{dx} = y^2 + 2xy\frac{dy}{dx} = 0 \qquad \text{with} \qquad 0 = 3y + 3x\frac{dy}{dx} + 4y\frac{dy}{dx}$$

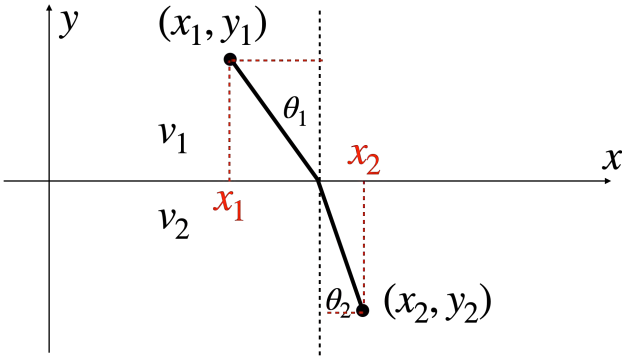
We want the ratio $r = x/y$, so rewrite the second equation as

$$3 + (3r + 4)\frac{dy}{dx} = 0 \qquad \text{so} \qquad \frac{dy}{dx} = -\frac{3}{3r+4}$$

and the first equation becomes

$$1 + 2r\left(-\frac{3}{3r+4}\right) = 0 \qquad \text{or} \qquad 3r + 4 - 6r = 0 \qquad \text{i.e.} \qquad r = \frac{4}{3}$$

(14) Refer to this annotated version of the figure:



The time it takes to get from point 1 to point 2 is

$$T(x) = \frac{1}{v_1}\left[(x-x_1)^2 + y_1^2\right]^{1/2} + \frac{1}{v_2}\left[(x_2-x)^2 + y_2^2\right]^{1/2}$$

In order to minimize $T(x)$ we need

$$\frac{dT}{dx} = \frac{1}{v_1}\frac{2(x-x_1)}{\left[(x-x_1)^2 + y_1^2\right]^{1/2}} - \frac{1}{v_2}\frac{2(x_2-x)}{\left[(x_2-x)^2 + y_2^2\right]^{1/2}} = 0$$

We recognize the ratios as the sines of the two angles. That is

$$\frac{1}{v_1}\sin\theta_1 - \frac{1}{v_2}\sin\theta_2 \qquad \text{or} \qquad \frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

In geometric optics, this is known as Snell's Law of Refraction. In that case, the speed of light is given by c/n in a medium with index of refraction n.

(15) To find the derivative of $\tan x$, just use the chain rule, that is

$$\frac{d}{dx}\frac{\sin x}{\cos x} = \left[\frac{\cos x}{\cos x} - \frac{\sin x\,(-\sin x)}{\cos^2 x}\right] = \frac{1}{\cos^2 x} = \sec^2 x$$

(16) $ax = \tan u$ means that $x = 0$ corresponds to $u = 0$ and $x = \infty$ to $u = \pi/2$. Also

$$\frac{1}{1+a^2x^2} = \frac{1}{1+\tan^2 u} = \frac{\cos^2 u}{\cos^2 u + \sin^2 u} = \cos^2 u$$

The integral then becomes simple. We have $a\,dx = du/\cos^2 u$ and

$$\int_0^\infty \frac{1}{1+a^2x^2}\,dx = \frac{1}{a}\int_0^{\pi/2} \cos^2 u \frac{du}{\cos^2 u} = \frac{\pi}{2a}$$

(17) The normal to the tangent line is just the line that goes from the center of the circle to the point $(\cos\theta, \sin\theta)$. The slope of this line is just $\Delta y/\Delta x = \sin\theta/\cos\theta = \tan\theta$.

(18) The length of the arc is

$$\int_{x=0}^{a} ds = \int_{x=0}^{a} \sqrt{dx^2 + dy^2} = \int_0^a \sqrt{1+y'^2}\,dx = \int_0^a \sqrt{1+\sinh^2 x}\,dx = \int_0^a \cosh(x)\,dx$$

which is just the same as the area under the curve.

(19) The equation of a circle is $x^2 + y^2 = R^2$ so $2x\,dx + 2y\,dy = 0$. Dividing through by $2dt$ tells us that $xv_x = -yv_y$. At the point $(x, y) = (R, 0)$ the x-component of the velocity is indeed zero, and at the point $(x, y) = (0, R)$ the y-component of the velocity is also zero. Most telling, though is when the particle is at 45° in the first quadrant, that is $(x, y) = (R/\sqrt{2}, R/\sqrt{2})$. Here, v_x and v_y are equal in magnitude but opposite in sign, and the relation still holds. Similarly in the other three quadrants.

(20) Newton's Second Law is $F = ma = mdv/dt$ and Hooke's Law is $F = -kx$ so

$$\frac{dE}{dt} = \frac{1}{2}m(2v)\frac{dv}{dt} + \frac{1}{2}k(2x)\frac{dx}{dt} = v\,m\frac{dv}{dt} + kx\,v = v\,F + (-F)\,v = 0$$

The time $T/4$ is the integral of t from $x = 0$ to $x = (2E/k)^{1/2}$, so

$$T = 4\int_{x=0}^{(2E/k)^{1/2}} dt = 4\int_0^{(2E/k)^{1/2}} \frac{dx}{v} = 4\int_0^{(2E/k)^{1/2}} \frac{dx}{\sqrt{(2/m)(E - kx^2/2)}}$$

Make the change of variables $x = (2E/k)^{1/2}\sin u$ in which case u runs from $u = 0$ to $u = \pi/2$. Also $E - kx^2/2 = E(1 - \sin^2 u) = E\cos^2 u$, and $dx = (2E/k)^{1/2}\cos u\,du$. Therefore

$$T = 4\sqrt{\frac{m}{2}}\sqrt{\frac{2E}{k}}\frac{1}{\sqrt{E}}\int_0^{\pi/2}\frac{\cos u\,du}{\cos u} = 4\sqrt{\frac{m}{k}}\frac{\pi}{2} = \frac{2\pi}{\sqrt{k/m}}$$

(21) The force dF on a horizontal strip of the dam is given by the product of the pressure and the area of the strip. The area of the strip is $w(y)dy$ where $y = 0$ at the lake's surface and $y = D$ at the bottom of the lake. We need $w(0) = W$ and $w(D) = 0$ where $w(y) = ay+b$ since the width varies linearly with depth. Therefore $b = W$ and $aD + b = aD + W = 0$ so $a = -W/D$ giving $w(y) = W(1 - y/D)$. Then

$$F = \int_0^D dF = \int_0^D p(y)w(y)dy = \rho gW\int_0^D \left(y - \frac{y^2}{D}\right)dy = \rho gW\left(\frac{D^2}{2} - \frac{1}{D}\frac{D^3}{3}\right) = \frac{\rho gWD^2}{6}$$

It is easy to see that this is dimensionally correct, since ρ has dimensions of mass per unit volume and WD^2 has dimensions of volume, so ρgWD^2 has dimensions of mass times acceleration, which is force.

In SI units, $\rho = 10^3$ kg/m^3, $g = 9.8$ m/s^2, and $WD^2 = 200 \times 160^2 = 5.1 \times 10^6$ m^3, so the total force is $10^3 \times 9.8 \times 5.1 \times 10^6/6 = 8.4 \times 10^9$ N=2×10^9 pounds or a million tons.

(22) Use $u = t/b$, then split the integral and use $f(1/a) = -f(a)$ to get

$$f(ab) = \int_1^{ab}\frac{1}{t}dt = \int_{1/b}^{a}\frac{1}{bu}bdu = \int_1^a\frac{1}{u}du + \int_{1/b}^1\frac{1}{u}du = f(a) - f(1/b) = f(a) + f(b)$$

Obviously, then, $f(a^2) = 2f(a)$ and so on for $f(a^n)$ for any positive integer n.

(23) Let y measure the vertical distance from the base to the tip. Then the radius $\rho(y)$ of a circular disk is $\rho(y) = r - (r/h)y = r(1 - y/h)$. The volume of a circular disk is $dV = \pi\rho^2 dy$ so the volume of the cone is

$$V = \int_0^h dV = \int_0^h \pi\rho^2 dy = \pi r^2 \int_0^h \left(1 - \frac{2}{h}y + \frac{1}{h^2}y^2\right) dy = \pi r^2 \left(h - \frac{h^2}{h} + \frac{h^3}{3h^2}\right) = \frac{1}{3}\pi r^2 h$$

To maximize the volume for fixed ℓ we write $V = \pi(\ell^2 - h^2)h/3$ and then find the derivative with respect to h. We have

$$\frac{dV}{dh} = \frac{\pi}{3}\frac{d}{dh}(\ell^2 - h^2)h = \frac{\pi}{3}(\ell^2 - 3h^2) = 0$$

Therefore the volume is a maximum when $h = \ell/\sqrt{3}$, or

$$\frac{h}{r} = \frac{h}{\sqrt{\ell^2 - h^2}} = \frac{1}{\sqrt{\ell^2/h^2 - 1}} = \frac{1}{\sqrt{3-1}} = \frac{1}{\sqrt{2}}$$

(24) See the accompanying MATHEMATICA notebook.

(25) See the accompanying MATHEMATICA notebook.

(26) See the accompanying MATHEMATICA notebook.

(27) See the MATHEMATICA notebook. The answers are

$$A = \frac{\sqrt{2\beta v_0 x_0 + v_0^2 + x_0^2(\beta^2 + \omega^2)}}{\omega} \qquad \text{and} \qquad \phi = -\cos^{-1}\left(\frac{x_0\omega}{\sqrt{2\beta v_0 x_0 + v_0^2 + x_0^2(\beta^2 + \omega^2)}}\right)$$

(28) I think it is easier to start with the right hand side and show that it equals the left.

$$\begin{aligned}
& \sinh(x)\cosh(y) + \sinh(y)\cosh(x) \\
= \; & \frac{e^x - e^{-x}}{2}\frac{e^y + e^{-y}}{2} + \frac{e^y - e^{-y}}{2}\frac{e^x + e^{-x}}{2} \\
= \; & \frac{e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}}{4} + \frac{e^{x+y} + e^{-x+y} - e^{x-y} - e^{-x-y}}{4} \\
= \; & \frac{e^{x+y} - e^{-x-y}}{2} = \sinh(x+y)
\end{aligned}$$

(29) Start with

$$I(a) = \int_{-\infty}^{\infty} e^{-ax^2}\, dx = \sqrt{\pi}a^{1/2} = \sqrt{\pi}a^{-1/2}$$

Now take the derivative with respect to a twice.

$$\begin{aligned}
I'(a) &= \int_{-\infty}^{\infty} (-x^2)e^{-ax^2}\, dx = \sqrt{\pi}a^{1/2} = -\frac{1}{2}\sqrt{\pi}a^{-3/2} \\
I''(a) &= \int_{-\infty}^{\infty} x^4 e^{-ax^2}\, dx = \sqrt{\pi}a^{1/2} = \frac{1}{2}\frac{3}{2}\sqrt{\pi}a^{-5/2}
\end{aligned}$$

The integral we want is

$$\int_0^\infty x^4 e^{-ax^2}\,dx = \frac{1}{2}I''(1) = \frac{3}{8}\sqrt{\pi}$$

(30) For (a), the mass density is $\lambda = m/L$, so $dm = \lambda dx = (m/L)dx$ and

$$x_{\rm CM} = \frac{1}{L}\int_0^L x\,dx = \frac{1}{L}\,\frac{x^2}{2}\bigg|_0^L = \frac{1}{L}\frac{L^2}{2} = \frac{1}{2}L$$

which is what you expect. For (b), we write $\lambda = ax$ and to determine a we have

$$m = \int_0^L \lambda\,dx = a\int_0^L x\,dx = a\frac{L^2}{2} \qquad \text{so} \qquad a = \frac{2m}{L^2}$$

The center of mass is therefore

$$x_{\rm CM} = \frac{1}{m}\int_0^L ax^2\,dx = \frac{2}{L^2}\,\frac{x^3}{3}\bigg|_0^L = \frac{2}{3}L$$

This makes sense. The rod gets heavier as you move to the right, so the center of mass should be somewhat right of center.

(31) Build the cone out of thin disks (which we analyzed in class) of radius $r(z)$ as shown in the figure, each of which has mass $dm = \rho \times \pi r^2\,dz$ and moment of inertia $dm\,r^2/2$. The mass density ρ is the cone mass m divided by its volume V. You can look up the volume of a circular cone, but it's easy enough to calculate, namely

$$V = \int_{\rm O} dV = \int_0^h \pi r^2\,dz = \pi\left(\frac{R}{h}\right)^2\int_0^h z^2\,dz = \pi\frac{R^2}{h^2}\,\frac{z^3}{3}\bigg|_0^h = \frac{1}{3}\pi R^2 h$$

Now calculate the moment of inertia by adding up the contributions from the little disks:

$$\begin{aligned} I &= \int_{\rm O}\frac{1}{2}dm\,r^2 = \frac{1}{2}\rho\pi\left(\frac{R}{h}\right)^4\int_0^h z^4\,dz = \frac{1}{2}\rho\pi\left(\frac{R}{h}\right)^4\frac{h^5}{5} \\ &= \frac{1}{2}\frac{m}{\pi R^2 h/3}\pi\frac{R^4 h}{5} = \frac{3}{10}mR^2 \end{aligned}$$

This calculation is also carried out in *Classical Mechanics*, by John Taylor, as Example 10.3, where the distinction is made that our I is the component I_{zz} of the inertia tensor.

(32) The calculation is straightforward:

$$I = \int_0^L dm\,r^2 = \int_0^L \lambda\,dr\,r^2 = \int_0^L \frac{m}{L}r^2\,dr = \frac{m}{L}\,\frac{r^3}{3}\bigg|_0^L = \frac{m}{L}\frac{L^3}{3} = \frac{1}{3}mL^2$$

(33) Build the annulus up from rings of density ρ and thickness dr, so

$$I = \int_{R_1}^{R_2} r^2\,dm = \int_{R_1}^{R_2} r^2\,2\pi r\,dr\,a\,\rho = 2\pi a\rho\int_{R_1}^{R_2} r^3\,dr = \frac{1}{2}\pi a\rho\left[R_2^4 - R_1^4\right]$$

To get the density, we need to know the volume V of the disk, so

$$V = \int_{R_1}^{R_2} 2\pi r\, dr\, a = 2\pi a \int_{R_1}^{R_2} r\, dr = \pi a \left[R_2^2 - R_1^2\right]$$

Since $M = \rho V$, we finally have

$$I = \frac{1}{2}\pi a \frac{M}{\pi a \left[R_2^2 - R_1^2\right]} \left[R_2^4 - R_1^4\right] = \frac{1}{2} M \left[R_2^2 + R_1^2\right]$$

which gives the correct answers in the limits $R_1 = 0$ and $R_1 = R_2$.