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1 Solutions for Chapter 1

1.1 Solutions to the Problems

Problem 1.1

You should find:

1. The Moon would be the size of an orange and its orbit around the Earth would barely fit into a basketball court.
2. The Sun would be the size of a basketball and the Earth's orbit would be about twice the basketball court. The most distant planet, Neptune, would be almost a kilometer away.
3. The Solar system would be the size of a grain of salt.
4. The Solar Neighbourhood would now be the size of a peppercorn.
5. The Milky Way would be the size of a small pancake (or a CD, for those who still remember what that is).
6. The Local Group would become the size of a watermelon.
7. The largest superclusters would be roughly the size of a cookie.

Problem 1.2

1. The Hubble time is

$$t_{H_0} \equiv \frac{1}{H_0} = \frac{\text{Mpc}}{68\,\text{km}\,\text{s}^{-1}} = \frac{10^6 \times 3.26 \times c \times \text{yrs}}{68\,\text{km}\,\text{s}^{-1}} = \frac{3.26 \times 10^6 \times 3 \times 10^5}{68}\,\text{yrs} \approx 14\,\text{billion yrs}\,. \tag{1.1}$$

   The universe is old.

2. The Hubble distance is

$$d_{H_0} \equiv \frac{c}{H_0} = 3 \times 10^8\,\text{m}\,\text{s}^{-1} \times t_{H_0} \approx 3 \times 10^8\,\text{m}\,\text{s}^{-1} \times 14 \times 10^9 \times (\pi \times 10^7)\,\text{s} \approx 10^{26}\,\text{m}\,. \tag{1.2}$$

   The universe is big.

3. The average density of the universe today is

$$\begin{aligned}\rho_0 = \frac{3H_0^2}{8\pi G} &= c^2 \times \frac{3}{d_{H_0}^2} \times \frac{1}{8\pi G} \\ &\approx (3\times 10^8\,\mathrm{m\,s^{-1}})^2\,\frac{3}{(10^{26}\,\mathrm{m})^2}\,\frac{1}{8\pi(6.67\times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}})} \\ &\approx 10^{-29}\,\mathrm{g\,cm^{-3}}\,. \end{aligned} \tag{1.3}$$

Compared to the density of water, $\rho_{\mathrm{H_2O}} \approx 1\,\mathrm{g\,cm^{-3}}$, we get

$$\frac{\rho_0}{\rho_{\mathrm{H_2O}}} = \frac{10^{-29}\,\mathrm{g\,cm^{-3}}}{1\,\mathrm{g\,cm^{-3}}} \approx 10^{-29}\,. \tag{1.4}$$

The universe is very dilute.

4. The number of atoms is

$$\begin{aligned} N_{\mathrm{H}} = \frac{\rho_0 V}{m_{\mathrm{H}}} &= \frac{10^{-29}\,\mathrm{g\,cm^{-3}}}{2\times 10^{-24}\,\mathrm{g}} \times \frac{4\pi}{3} d_{H_0}^3 \\ &\approx 2\times 10^{-5}\,\mathrm{cm^{-3}} \times (10^{28})^3\,\mathrm{cm^3} \\ &\approx 10^{79}\,. \end{aligned} \tag{1.5}$$

We assume that the density of the brain is that of water, $\rho_{\mathrm{brain}} \approx 1\,\mathrm{g\,cm^{-3}}$, and take the volume of the brain to be $V_{\mathrm{brain}} \approx 10^3\,\mathrm{cm^3}$. The number of hydrogen nuclei in the brain then is

$$\begin{aligned} N_{\mathrm{H}} = 2\times N_{\mathrm{H_2O}} &= 2\times \frac{\rho_{\mathrm{brain}}}{m_{\mathrm{H_2O}}}\,V_{\mathrm{brain}} \\ &= 2\times \frac{1\,\mathrm{g\,cm^{-3}}\times 10^3\,\mathrm{cm^3}}{3\times 10^{-23}\,\mathrm{g}} \\ &\approx 10^{26}\,. \end{aligned} \tag{1.6}$$

The ratio between the total number of hydrogen nuclei in the universe and those in your brain is

$$\frac{N_{\mathrm{universe}}}{N_{\mathrm{brain}}} \sim 10^{53}\,. \tag{1.7}$$

The universe contains a large number of particles.

5. The minimal distance probed by the LHC is

$$\ell_{\mathrm{min}} \equiv \frac{\hbar c}{E_{\mathrm{max}}} = \frac{0.2\times 10^{-6}\,\mathrm{eV\,m}}{10^{12}\,\mathrm{eV}} \approx 10^{-19}\,\mathrm{m}\,. \tag{1.8}$$

The ratio between the size of the observable universe and this length scale is

$$\frac{d_{H_0}}{\ell_{\mathrm{min}}} \sim 10^{45}\,. \tag{1.9}$$

Physics spans a large range of scales.

2 Solutions for Chapter 2

2.1 Solutions to the Exercises

Exercise 2.1

The Euler–Lagrange equation for the coordinate r is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \quad \Rightarrow \quad \ddot{r} = r\dot{\phi}^2 \,. \tag{2.1}$$

Similarly, for the coordinate ϕ, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) &= \frac{\partial L}{\partial \phi} \\ \frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\phi}) = 0 \quad &\Rightarrow \quad \ddot{\phi} = -\frac{2}{r}\,\dot{r}\dot{\phi} \,. \end{aligned} \tag{2.2}$$

This confirms the results given in the text.

Exercise 2.2

From the definition of the Christoffel symbol, we have

$$\Gamma^i_{0j} \equiv \frac{1}{2} g^{i\lambda}(\partial_0 g_{j\lambda} + \partial_j g_{0\lambda} - \partial_\lambda g_{0j}) \,. \tag{2.3}$$

Using $g^{i0} = g_{0j} = 0$ and $\partial_j g_{00} = 0$, this becomes

$$\begin{aligned} \Gamma^i_{0j} &= \frac{1}{2} g^{ik}\partial_0 g_{jk} \\ &= \frac{1}{2}(a^{-2}\gamma^{ik})c^{-1}\partial_t(a^2\gamma_{jk}) \\ &= c^{-1}\frac{\dot{a}}{a}\delta^i_j \,. \end{aligned} \tag{2.4}$$

Similarly, we have

$$\begin{aligned} \Gamma^i_{jk} &\equiv \frac{1}{2} g^{i\lambda}(\partial_j g_{k\lambda} + \partial_k g_{j\lambda} - \partial_\lambda g_{jk}) \\ &= \frac{1}{2} g^{il}(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}) \\ &= \frac{1}{2}\gamma^{il}(\partial_j \gamma_{kl} + \partial_k \gamma_{jl} - \partial_l \gamma_{jk}) \,. \end{aligned} \tag{2.5}$$

Exercise 2.3

In the text, we have shown that the geodesic equation of a particle with four-momentum $P^\mu = (E/c, P^i)$ can be written as

$$\frac{E}{c^3}\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{1}{c}\frac{\dot{a}}{a}p^2\,, \tag{2.6}$$

where $p^2 \equiv g_{ij}P^iP^j$. For massive particles, we have

$$g_{\mu\nu}P^\mu P^\nu = -c^{-2}E^2 + p^2 = -m^2c^2\,, \tag{2.7}$$

which implies that

$$\frac{E}{c^3}\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{p}{c}\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{1}{c}\frac{\dot{a}}{a}p^2\,, \tag{2.8}$$

and hence

$$\boxed{\frac{\dot{p}}{p} = -\frac{\dot{a}}{a}}\,. \tag{2.9}$$

The physical three-momentum of a massive particle therefore decreases as $p \propto a^{-1}$.

From the definition of the four-momentum of a massive particle, we get

$$P^i \equiv m\frac{\mathrm{d}x^i}{\mathrm{d}\tau} = m\frac{\mathrm{d}t}{\mathrm{d}\tau}\dot{x}^i = \frac{m\dot{x}^i}{\sqrt{1 - a^2\gamma_{ij}\dot{x}^i\dot{x}^j/c^2}} = \frac{m\dot{x}^i}{\sqrt{1 - v^2/c^2}}\,, \tag{2.10}$$

where we have introduced the *physical* peculiar velocity, $v^2 \equiv a^2\gamma_{ij}\dot{x}^i\dot{x}^j$. We therefore obtain the usual relativistic relation between momentum and velocity

$$\boxed{p = \frac{m\,v}{\sqrt{1 - v^2/c^2}}}\,. \tag{2.11}$$

Since p decreases with the expansion of the universe, so will v, showing that freely-falling particles will converge onto the Hubble flow.

Exercise 2.4

Using $U = (\rho c^2)V$, we have $\mathrm{d}U = c^2V(\mathrm{d}\rho + \rho\,\mathrm{d}V/V)$, so that $\mathrm{d}U = -P\mathrm{d}V$ becomes

$$\mathrm{d}\rho = -\frac{\mathrm{d}V}{V}\left(\rho + \frac{P}{c^2}\right) = -3\frac{\mathrm{d}a}{a}\left(\rho + \frac{P}{c^2}\right)\,, \tag{2.12}$$

where we used $V \propto a^3$ in the final equality. We therefore indeed get the continuity equation,

$$\boxed{\frac{\mathrm{d}\rho}{\mathrm{d}t} = -3H\left(\rho + \frac{P}{c^2}\right)}\,, \tag{2.13}$$

where $H \equiv \dot{a}/a$.

Exercise 2.5

Integrating the Friedmann equation, we have

$$H_0 t = \int_0^a \frac{\mathrm{d}a}{\sqrt{\Omega_r a^{-2} + \Omega_m a^{-1} + \Omega_\Lambda a^2 + \Omega_k}} \,. \tag{2.14}$$

Specializing to the case of a flat universe with only matter and radiation, this becomes

$$\begin{aligned} H_0 t &= \int_0^a \frac{\mathrm{d}a}{\sqrt{\Omega_r a^{-2} + \Omega_m a^{-1}}} \\ &= \frac{1}{\sqrt{\Omega_m}} \int_0^a \frac{\mathrm{d}a}{\sqrt{a_{\mathrm{eq}} a^{-2} + a^{-1}}} \\ &= \frac{1}{\sqrt{\Omega_m}} \frac{1}{\sqrt{a_{\mathrm{eq}}}} \int_0^a \frac{a\,\mathrm{d}a}{\sqrt{1 + (a/a_{\mathrm{eq}})}} \\ &= \frac{a_{\mathrm{eq}}^{3/2}}{\sqrt{\Omega_m}} \int_0^y \frac{y}{\sqrt{1+y}}\,\mathrm{d}y \\ &= \frac{a_{\mathrm{eq}}^{3/2}}{\sqrt{\Omega_m}} \int_0^y \left[\sqrt{1+y} - \frac{1}{\sqrt{1+y}}\right] \mathrm{d}y \\ &= \frac{a_{\mathrm{eq}}^{3/2}}{\sqrt{\Omega_m}} \left[\frac{4}{3} + \frac{2}{3}(1+y)^{3/2} - 2(1+y)^{1/2}\right], \end{aligned} \tag{2.15}$$

where $y \equiv a/a_{\mathrm{eq}}$. Using

$$H_0^{-1} \frac{a_{\mathrm{eq}}^{3/2}}{\sqrt{\Omega_m}} \approx 14.5 \times 10^9\,\mathrm{yrs} \times \frac{(2.9 \times 10^{-4})^{3/2}}{\sqrt{0.32}} \approx 130\,000\,\mathrm{yrs}\,, \tag{2.16}$$

we get

$$\boxed{t = 130\,000\,\mathrm{yrs}\left[\frac{4}{3} + \frac{2}{3}(1+y)^{3/2} - 2(1+y)^{1/2}\right]} \,. \tag{2.17}$$

For $y_{\mathrm{eq}} = 1$ and $y_{\mathrm{rec}} = 3400/1100$, this becomes

$$t_{\mathrm{eq}} \approx 50\,000\,\mathrm{yrs}\,, \tag{2.18}$$

$$t_{\mathrm{rec}} \approx 360\,000\,\mathrm{yrs}\,. \tag{2.19}$$

A similar analysis for a flat universe with only matter and dark energy gives

$$\begin{aligned} H_0 t &= \int_0^a \frac{\mathrm{d}a}{\sqrt{\Omega_m a^{-1} + \Omega_\Lambda a^2}} \\ &= \frac{a_{m\Lambda}^{3/2}}{\sqrt{\Omega_m}} \int_0^y \frac{\sqrt{y}}{\sqrt{1+y^3}}\,\mathrm{d}y \qquad y \equiv a/a_{m\Lambda} \\ &= \frac{a_{m\Lambda}^{3/2}}{\sqrt{\Omega_m}} \int_0^x \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x \qquad x \equiv y^{3/2} \end{aligned}$$

$$= \frac{2}{3}\frac{a_{m\Lambda}^{3/2}}{\sqrt{\Omega_m}}\sinh^{-1}(x)\,. \tag{2.20}$$

Using

$$\frac{2}{3}H_0^{-1}\frac{a_{m\Lambda}^{3/2}}{\sqrt{\Omega_m}} \approx \frac{2}{3}\times 14.5\times 10^9\,\text{yrs}\times\frac{0.77^{3/2}}{\sqrt{0.32}} \approx 11.1\,\text{billion yrs}\,, \tag{2.21}$$

this becomes

$$\boxed{t = 11.5\,\text{Gyrs}\times\sinh^{-1}(y^{3/2})}\,. \tag{2.22}$$

For $y_{m\Lambda}=1$ and $y_0 = 1/0.77$, we get

$$t_{m\Lambda} \approx 10.2\,\text{Gyrs}\,, \tag{2.23}$$

$$t_0 \approx 13.7\,\text{Gyrs}\,. \tag{2.24}$$

2.2 Solutions to the Problems

Problem 2.1

1. A nonzero g_{0i} is forbidden because it would introduce a preferred direction in space, breaking the isotropy assumption. A nontrivial $g_{00}(t)$ component can always be set to the constant -1 by redefining the time coordinate as $\mathrm{d}t' = \sqrt{-g_{00}(t)}\,\mathrm{d}t$. A spatial dependence in g_{00} is forbidden because this transformation would induce a nonzero g_{0i}.

2. We assume isotropy about a fixed point $\rho \equiv 0$ in spherical coordinates. At any fixed radial distance, we require rotational symmetry, so that all angular directions look the same. The spatial metric should then take the form

$$\mathrm{d}\ell^2 \equiv \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = A^2(\rho)\mathrm{d}\rho^2 + B^2(\rho)\mathrm{d}\Omega^2\,, \tag{2.25}$$

where $\mathrm{d}\Omega^2 \equiv \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$ is the metric on the unit 2-sphere. This amounts to foliating the spatial slices like an onion: the metric can depend on distance from the center of the onion, but we have rotational invariance along each layer at fixed ρ. Introducing a new radial variable $r \equiv B(\rho)$ puts the metric into its desired form

$$\mathrm{d}\ell^2 = e^{2\alpha(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{2.26}$$

where $e^{\alpha(r)} \equiv A(\rho(r))/B'(\rho(r))$.

Deriving the Ricci curvature is straightforward but tedious. The nonvanishing Christoffel symbols are

$$\begin{aligned}
&\Gamma^r_{rr} = \alpha'(r)\,, \quad \Gamma^r_{\theta\theta} = -re^{-2\alpha(r)}\,, \quad \Gamma^r_{\phi\phi} = -re^{-2\alpha}\sin^2\theta\,,\\
&\Gamma^\theta_{r\theta} = r^{-1}\,, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta\,,\\
&\Gamma^\phi_{r\phi} = r^{-1}\,, \quad \Gamma^\phi_{\theta\phi} = \cot\theta\,,
\end{aligned} \tag{2.27}$$

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